INFINITE IRREDUCIBLE SYSTEMS OF GROUP IDENTITIES

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Abstract

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INFINITE IRREDUCIBLE SYSTEMS OF GROUP IDENTITIES

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We shall call a system of group identical relations irreducible if each relation of this system does not follow from the remaining identities of the system.

The following theorem gives an effective solution of a well-known problem (see, for example, ([1]), p. 39).

Theorem 1. *For every odd natural number* $n \geqslant 4381$, *the set of all relations of the form*

$$(x^{rn}y^{rn}x^{-rn}y^{-rn})^n = 1, \tag{1}$$

where the parameter r runs through all prime numbers, is an irreducible system of group identities.

Theorem 1 immediately implies

Theorem 2. *There exists a continuum of distinct varieties of groups corresponding to subsets of the system of identities (1).*

The author is aware that recently A. Yu. Olshanskii obtained another proof of the continuum cardinality of the set of all varieties of groups, and that his proof does not give an effective indication of an infinite system of identities that is not equivalent to any finite system of identities.

Let l be some prime number. By $\mathfrak{M}_l$ we denote the set of all identical relations of type (1), where r runs through all prime numbers distinct from l.

To prove Theorem 1, a concrete group G_l with two generators is constructed in which all identities of the system $\mathfrak{M}_l$ are satisfied and the identity

$$(x^{ln}y^{ln}x^{-ln}y^{-ln})^n = 1 \tag{2}$$

is not satisfied.

We shall consider words in the two-letter group alphabet

$$a,\ b,\ a^{-1},\ b^{-1}. \tag{3}$$

In the work (2), in order to solve the well-known Burnside problem, the author, jointly with P. S. Novikov, constructed a theory of transformations of irreducible words in the alphabet (3) corresponding to the identical relation $x^n = 1$.

The construction of the required group G_l is based on an analogous theory corresponding to the system of identical relations $\mathfrak{M}_l$. The new theory will differ substantially from the original one, since in it it is necessary to exclude all transformations corresponding to the identity (2). For its construction, along with all the concepts of the work (2), induction on the rank α defines the concept of an admissible periodic word of rank $\alpha+1$. If, as a result of substituting into the left-hand side of relation (1), in place of the variables x and y, reduced words A and B of rank $\alpha \geqslant 0$ and interpreting the multiplication operation as concatenation of rank α, one obtains a word equivalent in rank α to the result of some concatenation of rank α of the form $[T, C^s, T^{-1}]_\alpha$, where $s > 0$, then we shall say that the word C is generated by substituting, in rank α, the pair of words (A, B) into relation (1). A periodic word C_1^{tC} of rank

a rank-$\alpha+1$ periodic word with period C minimal in rank α will be called admissible in rank α if one can indicate words A and B such that the word C, or some cyclic shift of it, is generated by substitution, in some rank $\beta \leqslant \alpha$, of the pair of words (A, B) into one of the relations of the system $\mathfrak{M}_l$.

In particular, the periodic word C^tC_1 of rank 1 with period C will be admissible in rank 0 if and only if some word of the form

$$A^{rn}B^{rn}A^{-rn}B^{-rn}, \tag{4}$$

where $r \neq l$, and A and B are words in the alphabet (3), is equal in the free group to some word of the form $TC^{sT^{-1}}$, where $s > 0$.

Whole, generated, and elementary words of rank $\alpha+1$ obtained from the initial periodic words of rank $\alpha+1$ admissible in rank α shall likewise be called admissible. Cyclic shifts of occurrences of admissible elementary words of rank $\alpha+1$ shall be called admissible cyclic shifts. We shall consider only admissible cyclic shifts of rank $\alpha+1$. In the definitions of cascade of rank α, real and really active occurrences of rank α of Ch. III of (2), only normalized occurrences of admissible elementary words of rank α must be considered. A normalized occurrence of an elementary word of rank α that is not admissible will not be really active, no matter how many sections of rank $\alpha-1$ it contains (see Lemmas 15 and 21 of Ch. III). Accordingly, in the definitions of the sets of words $N_\alpha, N'_\alpha, \Pi_\alpha, K_\alpha, K'_\alpha$, and M_α, the restrictions on the number of sections of rank $\alpha-1$ apply only to occurrences of admissible elementary words of rank α. In particular, words belonging to these classes may contain nuclei of rank α containing more than n

sections of rank $\alpha - 1$; but their bases will not be admissible elementary words of rank α, and these nuclei will not be really active.

The proofs of all assertions formulated in Chapters I–VI of (2) remain essentially the same; only some of these assertions must be reformulated in view of the remarks made above. In connection with Lemma 73 of Ch. V, let us note that the elementary p-powers of rank α which enter an admissible elementary p-power of rank $\alpha + 1$ may all be inadmissible. Such elementary words of rank $\alpha + 1$ will have type 1. It follows that if no admissible elementary p-power of rank $\beta \leqslant \alpha$ enters into an elementary word E of rank $\alpha + 1$, generated by some initial periodic word with period C, then E is a periodic word with period C. Lemma 73.2 of Ch. III is correct in the old formulation, but is useless for our purpose. If, however, the word "elementary" in it is replaced by "admissible elementary," then we obtain a false assertion. But it is easy to observe that the following analogue of this lemma will be true.

Lemma 1. If $X \overset{\alpha}{\sim} Y$ and no admissible periodic p-power of rank 1 enters the word X, then $X \overset{0}{=} Y$.

The required group G_l is constructed analogously to the way in which, in Chapter VII of (2), the group $\Gamma(2, n)$ was constructed. The assertion that all identities of the system $\mathfrak{M}_l$ hold in the group G_l is proved analogously to Lemma 22 of Ch. VII. However, this will not be a repetition of the proof of Lemma 22, and the following lemma will be needed, expressing one of the fundamental properties of the notion of admissibility introduced above.

Lemma 2. If the word C is generated by substitution, in rank $\alpha > 0$, of the pair of words (A, B) into the relation (1), where $r \neq l$, $C \in \Pi_{\alpha-1}$, and no admissible elementary $2p$-power of rank α enters C, then the word C can also be generated by substitution in rank $\alpha - 1$ of some pair of words (A_1, B_1) into the same relation (1).

Here it is appropriate to recall that every periodic word $C^t C_1$ of rank $\alpha+1$ with period C minimal in rank α is at the same time a periodic word of rank β with period C minimal in rank $\beta - 1$, for every

$0 < \beta \leq \alpha$. It follows from Lemma 2 that if the periodic word $C^t C_1$ of rank $\alpha+1$ is admissible in rank $\alpha > 0$, and no periodic $2p$-th power of rank 1 admissible in rank 0 occurs in the word C, then $C^t C_1$ is also admissible in rank 0.

From the definition of admissible periodic words of rank 1 and Lemma 2 one obtains

Lemma 3. *The word*

$$(a^{ln} b^{ln} a^{-ln} b^{-ln})^n \tag{5}$$

contains no admissible periodic p-th power of rank 1.

From Lemmas 1 and 3 and the definition of the group G_l it follows that the word (5) is not equal to the identity in G_l, i.e., the identical relation (2) is not satisfied in G_l.

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Note: Figure translations are in progress. See original paper for figures.