ON THE ASYMPTOTIC BEHAVIOR OF THE MAXIMUM OF GAUSSIAN RANDOM FIELDS

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Abstract

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MATHEMATICS

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ON THE ASYMPTOTIC BEHAVIOR OF THE MAXIMUM OF GAUSSIAN RANDOM FIELDS

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Let $\xi(t)$ be a stationary separable Gaussian real random process with spectral density $f(\lambda)$, satisfying the condition

$$\int_0^\infty \lambda^2[\ln(1+\lambda)]^a f(\lambda)\,d\lambda < \infty$$

for some $a > 1$.

T. Cramer (^{1}) proved that

$$\lim_{T\to\infty} P\left\{\left|\max_{0\le t\le T}\xi(t) - \sqrt{2\ln T}\right| < \frac{\ln\ln T}{\sqrt{2\ln T}}\right\} = 1.$$

M. G. Shur (^{2}) established that, under the same assumptions, with probability one

$$\max_{0\le t\le T}\xi(t) - \sqrt{2\ln T} \longrightarrow 0.$$

In the present note the asymptotic behavior of the maximum of random fields is investigated. The methods of proof differ from (^{1},^{2}), since in the case of functions of several variables one cannot use estimates connected with the number of level crossings.

Consider a separable real Gaussian homogeneous isotropic random field $\xi(\bar{x})$ in n-dimensional Euclidean space R^n. As is known,

$$M\xi(\bar{x}_1)\xi(\bar{x}_2) = R_1(\bar{x}_1 - \bar{x}_2) = R_2(r),$$

where $r^2 = (\bar{x}_1 - \bar{x}_2)(\bar{x}_1 - \bar{x}_2)'$. Without loss of generality, we shall assume that $M\xi(\bar{x}) = 0,\ D\xi(\bar{x}) = 1$.

Let ν be Lebesgue measure defined on the σ-algebra $\mathfrak{M}$ of Lebesgue-measurable sets in R^n. Consider in R^n the parallelepiped $D(X_1, \dots, X_n) = (0 \le x_1 \le X_1, \dots, 0 \le x_n \le X_n)$ and a measurable closed simply connected domain E_1, bounded by the surface Φ_1, $\nu(E_1) = 1,\ \bar{0} \in E_1$.

Make a similarity transformation of the surface Φ_1 with center at the origin and similarity coefficient $K > 1$; we obtain the surface Φ_k, bounding the domain E_k.

Theorem. Suppose that, with probability one, there exist continuous first and second partial derivatives of the random field $\xi(\bar{x})$, and

$$P\{\det \|\partial^2 \xi(\bar{x}) / \partial x_i \partial x_j\| = 0\} = 0.$$

Let the correlation function of the field satisfy the conditions:

$$\text{I.} \quad \left| \frac{\partial^4 R_1(\bar{x})}{\partial x_1^{\varepsilon_1} \cdots \partial x_n^{\varepsilon_n}} - \frac{\partial^4 R_1(\bar{0})}{\partial x_1^{\varepsilon_1} \cdots \partial x_n^{\varepsilon_n}} \right| < N_1 \sum_{i=1}^{n} |x_i|^{\delta},$$

$$\varepsilon_i = 0, 2, 4; \quad N_1 = \text{const} > 0; \quad \sum_{i=1}^{n} \varepsilon_i = 4; \quad \delta > 0.$$

$$\text{II.} \quad |R_2(r)| < N_2 / r^n, \quad N_2 = \text{const}.$$

Then for every $\varepsilon > 0$, almost surely there exist random $K^0(\varepsilon) < \infty,\ X_i^0(\varepsilon) < \infty$ $(i = 1, \dots, n)$, such that for $K > K^0(\varepsilon)$

$$\left| \max_{\bar{x} \in E_k} \xi(\bar{x}) - \sqrt{2 \ln \nu(E_k)} \right| < (2 + \varepsilon) \frac{\ln \ln \nu(E_k)}{\sqrt{2 \ln \nu(E_k)}},$$

and for $X_i > X_i^0(\varepsilon)$, $i = 1, \dots, n$,

$$\left| \max_{\bar{x} \in D(X_1, \dots, X_n)} \xi(\bar{x}) - \left(2 \ln \prod_{i=1}^{n} X_i \right)^{1/2} \right| < (n{+}1{+}\varepsilon) \ln \ln \prod_{i=1}^{n} X_i \Big/ \left(2 \ln \prod_{i=1}^{n} X_i \right)^{1/2}.$$

From the work of Yu. K. Belyaev (^{3}) it follows that, under the conditions of the theorem, there exists μ_c, the intensity of the mean number of local maxima of the random field exceeding the level c.

Lemma 1. For arbitrary $c > 0$,

$$\mu_c \le f(n) e^{-c^2/2},$$

where $f(n)$ is finite for every n and does not depend on c.

Lemma 2. For every $c > 0$, $k > 1$,

$$P\left\{\max_{\bar{x} \in E_k} \xi(\bar{x}) > c\right\} \le N_3 \nu(E_k) e^{-c^2/2}. \tag{1}$$

Moreover, if $X_i > 1$ $(i = 1, \dots, n)$, then

$$P\left\{\max_{\bar{x} \in D(X_1, \dots, X_n)} \xi(\bar{x}) > c\right\} \le N_3 \prod_{i=1}^{n} X_i e^{-c^2/2}. \tag{2}$$

Here N_3 is a constant that does not depend on $c, k, X_1, \dots, X_n$.

Lemma 3. Under the conditions of the theorem, for every $c > 0$,

$$P\left\{\max_{\bar{x} \in G} \xi(\bar{x}) < c\right\} \le N_4 c^2 \left(e^{c^2/2} + e^{c^2/3} \ln \nu(G)\right) / \nu(G), \tag{3}$$

where G coincides with $D(X_1, \dots, X_n)$ or with E_k; N_4 does not depend on $c, X_1, \dots, X_n, k$, and $X_i > 1$ $(i = 1, \dots, n)$, $k > 1$.

Lemma 4. Suppose the following series converge:

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{m} P\left\{\max_{\bar{x} \in D(e^{m_1}, \dots, e^{m_n})} \xi(\bar{x}) >\right.$$

$$\left. > \left[2\sum_{i=1}^{n} m_i + \left(\ln \sum_{i=1}^{n} m_i\right)(n + 1 + \varepsilon)\right] \Big/ \left(2\sum_{i=1}^{n} m_i\right)^{1/2}\right\},$$

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} P\left\{\max_{\bar{x} \in D(e^{m_1}, \dots, e^{m_n})} \xi(\bar{x}) <\right.$$

$$\left. < \left[2\sum_{i=1}^{n} m_i - \left(\ln \sum_{i=1}^{n} m_i\right)(n + 1 + \varepsilon)\right] \Big/ \left(2\sum_{i=1}^{n} m_i\right)^{1/2}\right\},$$

$$\sum_{m=1}^{\infty} P\left\{\max_{\bar{x} \in E_{em}} \xi(\bar{x}) > (2mn)^{1/2} + (2 + \varepsilon)\ln(mn)/(2mn)^{1/2}\right\},$$

$$\sum_{m=1}^{\infty} P\left\{\max_{\bar{x}\in E_{e^m}} \xi(\bar{x}) < \sqrt{2mn} - (2+\varepsilon)\ln(mn)/(2mn)^{1/2}\right\}.$$

Then the assertion of the theorem holds.

The convergence of the series can be verified by substituting the corresponding values for c in (1), (2), (3).

In conclusion, I express my sincere gratitude to M. I. Yadrenko for posing the problem and for his attention to this work.

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Note: Figure translations are in progress. See original paper for figures.