ON NECESSARY AND SUFFICIENT CONDITIONS FOR POINTS OF THE MARKOV SPECTRUM TO BELONG TO THE LAGRANGE SPECTRUM

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Abstract

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ON NECESSARY AND SUFFICIENT CONDITIONS FOR POINTS OF THE MARKOV SPECTRUM TO BELONG TO THE LAGRANGE SPECTRUM

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The present paper is a development of the ideas and methods of the work of G. A. Freiman ([1]).

Notation: x is a sequence of natural numbers $a_0, \dots, a_i, \dots$;

$$[q_1; q_2, \dots] = q_1 + \frac{1}{q_2 + \dots}; \qquad \lambda_i(\mathcal{L}) = a_i + [0; a_{i+1}, \dots] + [0; a_{i-1}, \dots, a_0];$$

$$\lambda(\mathcal{L}) = \varlimsup_{i\to\infty} \lambda_i(\mathcal{L});$$

$\{\lambda(\mathcal{L})\}$ is the Lagrange spectrum; M is a sequence of natural numbers infinite in both directions $\dots, a_{-1}, a_0, a_1, \dots$;

$$\lambda_i(M) = a_i + \delta_i + \gamma_i,$$

where

$$\delta_i = [0; a_{i+1}, a_{i+2}, \dots]; \qquad \gamma_i = [0; a_{i-1}, a_{i-2}, \dots]; \qquad \lambda(M) = \sup_i \lambda_i(M);$$

$\{\lambda(M)\}$ is the Markov spectrum;

$$\alpha = 4\sqrt{30}/7, \qquad \beta = \sqrt{689}/8.$$

A sequence $M_1, M_2, \dots, M_j, \dots$ will be called **convergent** to the sequence M if, for any integer i, we have

$$\lim_{j\to\infty} a_i^{(j)} = a_i, \qquad \lim_{j\to\infty} \lambda(M_j) = \lambda(M).$$

A sequence $\{M_j\}$, $j = 1, 2, \ldots$, will be called **stabilizing from the right** (**from the left**) if there exist a natural number j_0 and an integer i_0 such that, for $i > i_0$ and $j > j_0$ (respectively $i < i_0$),

$$a_i^{(j)} = a_i. \tag{1}$$

Theorem 1. *Let $\lambda(M) \in (\alpha, \beta)$. If there exists a sequence $\{M_j\}$ converging to M and not stabilizing either from the right or from the left, then $\lambda(M) \in \{\lambda(\mathcal{L})\}$.*

Proof. Without loss of generality, we assume that $\lambda(M) = \lambda_0(M)$. Let, for the given j, condition (1) be satisfied for

$$-i_0 \le i \le i_1; \qquad i_0, i_1 > 0,$$

$$a^{(j)}_{-i_0-1} \ne a_{-i_0-1}, \qquad a^{(j)}_{i_1+1} \ne a_{i_1+1}.$$

Define the sequence

$$\bar{M}_j = \{\ldots, \bar{a}_{-1}, \bar{a}_0, \bar{a}_1, \ldots\}$$

by the equalities:

$$\bar{a}_i = a_i, \quad -i_0 \le i \le i_1; \qquad \bar{a}_{i_1+1} = \bar{a}_{i_1+2} = 2; \qquad \bar{a}_{i_1+3} = 1; \qquad \bar{a}_{i_1+4} = \bar{a}_{i_1+5} = \bar{a}_{i_1+6} = 2;$$

$$\bar{a}_i = \bar{a}_{i-4}, \quad i \ge i_1 + 6; \qquad \bar{a}_{-i_0-1} = \bar{a}_{-i_0-2} = 2; \qquad \bar{a}_{-i_0-3} = 1;$$

$$\bar{a}_{-i_0-4} = \bar{a}_{-i_0-5} = \bar{a}_{-i_0-6} = 2; \qquad \bar{a}_i = \bar{a}_{i+4}, \quad i \le -i_0 - 6,$$

i.e.

$$\bar{M}_j = \{\ldots, 2, 2, 2, 1, 2, 2, a_{-i_0}, \ldots, a_{i_1}, 2, 2, 1, 2, 2, 2, \ldots\}.$$

For this sequence, when $i > i_1$, $i < -i_0$, we have

$$\lambda_i(\bar{M}_j) \le \alpha, \tag{2}$$

since for the combinations

$$\begin{Bmatrix} 222 \\ i \end{Bmatrix}, \quad \begin{Bmatrix} 122 \\ i \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} 212 \\ i \end{Bmatrix}$$

we obtain the estimate $\lambda_i \le \alpha$, if one takes into account that the presence in M of the combinations $\{3\}$ and $\{1212\}$ gives $\lambda(M) > \beta$.

Let now $-i_0 \le i \le i_1$. We shall show that

$$\delta_i(\overline{M}_j) \le \max(\delta_i(M),\ \delta_i(M_j)), \tag{3}$$

$$\gamma_i(\overline{M}_j) \le \max(\gamma_i(M),\ \gamma_i(M_j)). \tag{4}$$

We shall prove inequality (3), since (4) is proved analogously. Let i have the same parity as i_1-1. Suppose, for example, that $a_{i_1+1}=2,\ a^{(j)}_{i_1+1}=1$. Then

$$[0;\ a_i,\ \overline{a}_{i+1},\dots,a_{i_1},\ 2,\dots]<[0;\ a_i,\dots,a_{i_1},\ 1,\dots]$$

for any $a_i,\ i<i_1$. Therefore

$$\delta_i(\overline{M}_j)\le\delta_i(M_j). \tag{5}$$

Let now i have the same parity as i_1. If $a_{i_1+2}=1$, then

$$[0;\ a_i,\dots,a_{i_1},\ 2,\ 2,\dots]<[0;\ a_i,\dots,a_{i_1},\ 2,\ 1,\dots]. \tag{6}$$

If $a_{i_1+2}=2,\ a_{i_1+3}=2$, then

$$[0;\ a_i,\dots,a_{i_1},\ 2,\ 2,\ 1,\dots]<[0;\ a_i,\dots,a_{i_1},\ 2,\ 2,\ 2,\dots]. \tag{7}$$

If $a_{i_1+2}=2,\ a_{i_1+3}=1,\ a_{i_1+4}=1$, then

$$[0;\ a_i,\dots,a_{i_1},\ 2,\ 2,\ 1,\ 2,\dots]<[0;\ a_i,\dots,a_{i_1},\ 2,\ 2,\ 1,\ 1,\dots]. \tag{8}$$

If $a_{i_1+2}=2,\ a_{i_1+3}=1,\ a_{i_1+4}=a_{i_1+5}=2$, then $a_{i_1+5}=2$, since otherwise the combination $\{\overset{2121}{i}\}$ would occur in M. If $a_{i_1+2}=2,\ a_{i_1+3}=1,\ a_{i_1+4}=a_{i_1+5}=2$ and $a_{i_1+6}=1$, then

$$[0;\ a_i,\dots,a_{i_1},\ 2,\ 2,\ 1,\ 2,\ 2,\ 2,\dots]<[0;\ a_i,\dots,a_{i_1},\ 2,\ 2,\ 1,\ 2,\ 2,\ 1,\dots]. \tag{9}$$

Each of the inequalities (6), (7), (8), (9) entails the validity of the inequality

$$\delta_i(\overline{M}_j)<\delta_i(M).$$

Since the length of the period in $\overline{M}_j$ is equal to 4, inequality (3) is proved completely.

From (1)–(4) it follows that

$$\lim_{j\to\infty}\lambda(\overline{M}_j)=\lambda(M). \tag{10}$$

Let

$$T_j,\ R_j \geq \max(2i_0,\ 2i_1), \tag{11}$$

with $a^{(j)}_{R_j} = 1$ and $\overline{a}^{(j)}_{-T_j} = \overline{a}^{(j)}_{-T_j+1} = \overline{a}^{(j)}_{-T_j+2} = 2$.

For the sequence

$$\mathcal{L} = \{a^{(1)}_{-T_1}, \dots, a^{(1)}_{R_1},\ a^{(2)}_{-T_2}, \dots, a^{(2)}_{R_2}, \dots\}$$

in view of (10) and (11),

$$\lambda(\mathcal{L}) = \lambda(M).$$

A direct consequence of Theorem 1 is

Theorem 2. Let $\lambda(M) \in (\alpha, \beta)$, and suppose there exists an infinite set of sequences M' for which $\lambda(M') = \lambda(M)$. Then $\lambda(M) \in \{\lambda(\mathcal{L})\}$.

We say that for some $\varepsilon > 0$ the sequence M has the ε-property on the right (on the left) if there exists an integer i_0 such that for any sequence $M' \neq M$ for which $a_i = a'_i$ for $i < i_0$ (respectively $i > i_0$), we have $\lambda(M') > \lambda(M) + \varepsilon$. If $\lim\limits_{i \to +\infty} \lambda_i(M) < \lambda(M)$, then the sequence is called indefinite.

Theorem 3. Let $\lambda(M_1) \in (\alpha, \beta)$. In order that $\lambda(M_1) \in \overline{\{\lambda(\mathcal{L})\}}$, the following conditions are necessary and sufficient:

1. There exists only a finite number of sequences $\overline{M_s}$, $1 \leq s \leq k$, such that
$$\lambda(\overline{M_1}) = \lambda(\overline{M_2}) = \dots = \lambda(\overline{M_k}).$$
2. The sequences M_s are nonperiodic.
3. The sequences M_s are periodic on the right*, i.e., there exist an integer i_0 and a natural number p such that, for $i > i_0$, $a_i = a_{i+p}$.
4. Each of the sequences M_s, periodic in both directions, with period equal to the period of the sequence M_s, has the ε-property on the right, where
$$\varepsilon = \lambda(\overline{M_s}) - \lim_{i \to \infty} \lambda_i(M_s).$$

Proof. The necessity of condition 1 follows from Theorem 2. Since the sequence M_1 is nonperiodic (see (2)), there exist an integer i_1 and $\varepsilon_1 > 0$ such that $\lambda_i(M_1) < \lambda(M_1) - \varepsilon_1$ for $i > i_1$. One can choose an $s = s(\varepsilon_1)$ such that if in M_1 and M' we have $a_i = a'_i$ for $i \leq j + s$ or $i \geq j - s$, then

$$\lambda_j(M') < \lambda_j(M) + \frac{\varepsilon_1}{2}. \tag{12}$$

There will be natural numbers $b_1, b_2, \ldots, b_{2s}$ and sequences $i_1, i_2, \ldots, i_r, \ldots$ such that $a_{i_r+t} = b_{t+1}$, $t = 0, 1, \ldots, 2s-1$, $r = 1, 2, \ldots$. Form the sequences

$$M^{(r)} = \{a_i^{(r)} = a_i,\ i \le i_r,\ a_i^{(r)} = a_{i+i_1-i_r},\ i > i_r\}. \tag{13}$$

For them, in view of (12) and (13),

$$\lambda_i(M^{(r)}) < \lambda_i(M_1) + \frac{\varepsilon_1}{2}, \quad i \le i_r + s.$$

In view of (12),

$$\lambda_i(M^{(r)}) < \lambda(M_1), \quad i > i_r + s.$$

Analogous arguments can also be carried out for small i. If we assume that M_1 is nonperiodic both on the right and on the left, then it would now follow from Theorem 1 that $\lambda(M_1) \in \overline{\{\lambda(\mathcal{L})\}}$.

Finally, condition 4 is also satisfied, since otherwise it would follow from Theorem 1 that $\lambda(M_1) \in \overline{\{\lambda(\mathcal{L})\}}$.

Conditions 1-4 are also sufficient in order that $\lambda(M_1) \in \overline{\{\lambda(\mathcal{L})\}}$. Indeed, if $\lambda(M_1) = \lambda(\mathcal{L})$ for some $\mathcal{L}$, then in $\mathcal{L}$ there would occur arbitrarily long segments of the sequences $M^{(r)}$, and this contradicts conditions 1-4.

All the theorems of the present paper are formulated only for $\lambda(M) \in (\alpha, \beta)$. The set for which Theorems 1-3 are valid can be substantially enlarged.

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1. G. A. Freiman, *Mathematical Notes*, **3**, No. 2, 195 (1968).

2. P. G. Kogonia, *Proceedings of the Tbilisi Mathematical Institute*, **29**, 15 (1963).

* The sequence M can always be replaced by M', for which $b_i = a_{-i}$, and we shall not distinguish between two such sequences.

Note: Figure translations are in progress. See original paper for figures.