SOME PROPERTIES OF PLANE AND SPATIAL MAPPINGS

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SOME PROPERTIES OF PLANE AND SPATIAL MAPPINGS

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1. In this note we formulate results established on the basis of inequality (1) from the work ([1]).

We shall consider a family $\{f\}$ of homeomorphic mappings $y = f(x)$, defined in a domain D of n-dimensional Euclidean space E^n (unless the contrary is stipulated). Denote

$$I(f, D, F) = \int_D F\left(x, f, \frac{df}{dx}\right) dx \tag{1}$$

the functional defined on the mappings of the family $\{f\}$, where in (1) $x = (x_1, \dots, x_n)$, $f = (f_1(x), \dots, f_n(x))$, $df/dx = (\partial f_i/\partial x_j)$ is an $n \times n$-matrix whose elements are partial derivatives understood in the sense of S. L. Sobolev, $F(x, y, Z)$ is a measurable function of its $2n + n^2$ arguments, and $Z = (z_{ij})$ is an $n \times n$-matrix.

Below the following notation will be used: $\rho(M_1, M_2)$ is the distance between sets in E^n; $|x' - x''|$ is the distance between points in E^n; $\overline{M}$ is the closure of the set M in E^n; ∂D is the boundary of the domain D in E^n; $d(M)$ is the diameter of the set M in E^n; $\widetilde{E}^n$ is the completion of n-dimensional space with respect to the spherical metric $\tilde{\rho}(x', x'')$; $d_\sim(M)$ is the diameter of the set M in the metric $\tilde{\rho}$. Suppose that in the domain D a positive continuous function $h(x)$ is given, which generates in D a non-Euclidean metric with line element $ds = h(x)\, dl$, where dl is the line element in E^n. By $\rho_h(M_1, M_2)$ we shall denote the distance between the sets M_1 and M_2 in this non-Euclidean metric, and by $d_h(M)$ the diameter of the set M in this metric.

2. **Theorem 1.** *Let, for a family of mappings $\{f\}$ of a domain D onto a domain Δ, the functional*

$$I(f, \Delta^{-1}, \Phi) = \int_\Delta \Phi\left(y, f^{-1}, \frac{df^{-1}}{dy}\right) dy \le K,$$

be bounded, where $\Phi(y, x, Z) \geq h^n(x)\|Z\|^n$, *the function* $h(x)$ *is defined, continuous, and positive in* D,

$$\|Z\| = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} z_{ij}^2\right)^{1/2}.$$

Assume that there exist a continuum $H \subset D$ *and a number* $a > 0$ *such that*

$$d(f(H)) > 2\rho(f(H), \partial\Delta), \qquad d(f(H)) \geq a, \tag{2}$$

for all $f \in \{f\}$. *Then*

$$\rho(f(H), \partial\Delta) \geq \frac{\alpha}{2}\exp[-M_{nK}\rho_h^{-n}(H, \partial D)], \tag{3}$$

where M_n *is an absolute constant* ($M_n > 0$).

Remark. Inequality (3) shows that, for any point $a \in H$, under each mapping $f \in \{f\}$ the ball of radius $\frac{1}{2}\alpha\exp[-M_{nK}\rho_h^{-n}(H, \partial D)]$ with center at the point $f(a)$ lies entirely in the domain Δ.

We shall give, for the planar case, some sufficient conditions for the fulfillment of relations (2).

a) Let the domain D be simply connected, and suppose that on the boundaries of the domains D and Δ three distinct prime ends e_i and e_i^* are fixed, with $f(e_i) = e_i^*$ for each $f \in \{f\}$. Let a be a certain principal point of the prime end e_1. Suppose that there exists a domain $U_1 \subset D$, into which the prime end e_1 enters, and a neighborhood U_2 of the point a in E^2, such that for all $f \in \{f\}$ the inequality

$$I(f, U_3, F) = \int_{U_3} F\left(x, f, \frac{df}{dx}\right) dx \leqslant K < \infty, \tag{4}$$

holds, where $U_3 = U_1 \cap U_2$, $F(x, y, Z) \geqslant h^2(y)\|Z\|^2$, and $h(y)$ is a continuous function in E^2 generating a metric ρ_h topologically equivalent to the spherical metric. Suppose that an inequality analogous to (4) is also satisfied for the prime end e_2. Then there exists a continuum H for which the inequalities (2) are satisfied.

b) Suppose that on the boundaries of the domains D and Δ three distinct prime ends e_i and e_i^* $(i = 1, 2, 3)$ are fixed, and that an open arc of prime ends e_1e_2, not containing e_3, is carried under each mapping $f \in \{f\}$ into the arc of prime ends $e_1^*e_2^*$, not containing the prime end e_3^*. Suppose that $f(e_3) = e_3^*$ for all $f \in \{f\}$. Let there be in the domain D a curve $L : x = \pi(t)$ $(0 < t < 1)$. Put $E_\varepsilon = [\varepsilon, 1-\varepsilon]$ $(0 < \varepsilon < 1/2)$. Suppose that $f(\pi(t))$ tends uniformly with respect to $f \in \{f\}$ to e_3^* as $t \to 1$, and that $f(\pi(t))$ tends uniformly with respect to $f \in \{f\}$ to the arc of prime ends $e_1^* e_2^*$ as $t \to 0$. Then, for sufficiently small ε, the continuum $\pi(E_\varepsilon)$ will satisfy the inequalities (2).

3. **Theorem 2.** *Let there be given a family $\{f\}$ of quasiconformal mappings, each of which is a homeomorphism of the domain $D \subset E^2$ onto the strip $\Delta = \{y : 0 < y_2 < \delta\}$. Suppose that on the boundaries of the domains D and Δ three distinct prime ends e_i and e_i^* $(i = 1, 2, 3)$ are fixed, the bodies of the prime ends e_1^* and e_2^* being ∞, and the body of the prime end e_3^* being a certain point $a \in \partial\Delta$. Suppose that $f(e_i) = e_i^*$ $(i = 1, 2, 3)$ for all $f \in \{f\}$. Suppose that for all $f \in \{f\}$ the inequality*

$$I(f^{-1}, \Delta, \Phi) \leqslant K,$$

is satisfied, where $\Phi(y, x, Z) = h^2(x)\|Z\|^2$, and $h(x)$ is a continuous positive function in the domain D, and moreover such that in certain subdomains g_i $(i = 1, 2, 3)$ of D, into each of which one prime end e_i $(i = 1, 2, 3)$ enters, this function generates a metric ρ_h topologically equivalent to the spherical metric. Consider a domain $G \subset D$ having the property that the set $D \setminus G$ consists of two subdomains, with the prime end e_1 entering one of them and e_2 the other, while the prime end e_3 enters the domain G. Then

$$\sup_{x \in G} |f(x) - a| \leqslant \delta \exp\left[2\pi K \beta^{-2}(G)\right]. \tag{5}$$

Here $\beta(F) = \inf d_h(l)$, and the infimum is taken over all curves $l \subset D$ such that l separates e_3 from e_1 or from e_2, and the set $l \cap G$ is nonempty.

Let us note that, when the hypotheses of this theorem are satisfied, $\beta(G) > 0$, and inequality (5) gives a nontrivial estimate for the order of growth up to the boundary for the family of mappings $\{f\}$.

4. **Theorem 3.** *Suppose that $y = f(x)$ is a homeomorphic mapping of a simply connected domain $D \subset E^2$ onto a domain Δ. Let the boundary of the domain Δ contain a rectilinear segment γ^* (finite or infinite), lying on a certain line l parallel to the axis Oy_1, and let the domain Δ lie on one side of the line l. Let some arc of prime ends γ*

of the domain D under the mapping $y = f(x)$ passes into γ^*, and let the integral be bounded

$$\int_D |\nabla f_2|^2\, dx \leqslant K,$$

where the derivatives are understood in the sense of S. L. Sobolev. Consider a family $\{S_r\}$ of concentric circles S_r of radii r such that the set $S_r' = S_r \cap D$ is nonempty for $r \in (0, r_2)$. Let a component K_r be chosen from each set S_r', in such a way that the ends of the arc K_r lie on the arc γ, and the family $\{K_r\}$ determines some prime end $e \in \gamma^*$. Denote by

$$\alpha(r) = \sup_{y \in f(K_r)} \rho(y, \gamma^*)$$

the deviation of the set $f(K_r)$ from the set γ^*, and suppose that $\alpha(r)$ is a measurable function.

Then the inequality

$$\int_{r_1}^{r_2} \frac{\alpha^2(r)}{r}\, dr \leqslant 2\pi K. \tag{6}$$

holds.

5. Theorem 4. Let $y = f(x)$ be a homeomorphic mapping of a simply connected domain $D \subset E^2$ onto a domain Δ lying in the strip $G = \{y : \delta_1 < y_2 < \delta_2\}$ ($\delta_i = \text{const}$). Let two prime ends e_1 and e_2, dividing the boundary of the domain D into two open arcs of prime ends γ_1 and γ_2, be taken on the boundary of the domain D. Denote $\gamma_i^* = \{y : y_2 = \delta_i\}$ $(i = 1, 2)$, and suppose that for every prime end $e \in \gamma_i$, $\rho(f(x), \gamma_i^*) \to 0$ as $x \to e$. Let the integral

$$\int_g \frac{1}{(1 + |f|^2)^2} |\nabla f_1|^2\, dx, \tag{7}$$

be bounded, where g is some subdomain of D, into which the prime end e_1 enters, and $q = \partial g \setminus \partial D$ is a section of the domain D. Denote

$$G_{-\delta} = \{y \in G : y_2 \leqslant -\delta\}, \quad G_\delta = \{y \in G : y_2 \geqslant \delta\}, \quad R_\delta = \{y \in G : y_1 = \delta\}.$$

Then two cases are possible: 1) there exists a number $E > 0$ such that either $G_{-E} \subset \Delta$, or $G_E \subset \Delta$; 2) there exists a number E such that either $G_{-E} \subset G \setminus \Delta$ and $R_{-E} \subset \partial\Delta$, or $G_E \subset G \setminus \Delta$ and $R_E \subset \partial\Delta$.

Suppose now that, instead of boundedness of the integral (7), boundedness of the integral

$$\int_g \frac{1}{(1 + |f|^2)^2} |\nabla f_2|^2\, dx$$

holds.

Denote by G_1 the component of the set $G \setminus f(q)$ containing the set $f(g)$. Then the set $f(g)$ may be obtained from the set G_1 by making a finite or countable number of cuts in G_1, going from infinity parallel to the axis Oy_1.

Remark. For a conformal mapping $y = f(x)$, under the hypotheses of Theorem 4 the domain Δ will always coincide with the strip G. The following example shows that case 2) in Theorem 4 can actually occur. Let

$$D = \{x \in E^2 : 0 < x_2 < 1\}, \quad \Delta = \{y \in E^2 : 0 < y_1,\ 0 < y_2 < 1\},$$

$$y_1 = f_1(x) \equiv e^{x_1}, \quad y_2 = f_2(x) \equiv x_2.$$

In this case all the hypotheses of Theorem 4 are fulfilled, and the integral (7) is bounded for $g = D_{-E}$, $E = 1$.

Theorem 5. Suppose that $y = f(x)$ is a homeomorphic mapping of a simply connected domain $D \subset E^2$ onto a domain Δ. Let $\{S_r\}$ be a family of concentric circles such that the set $S'_r = S_r \cap D$ is not

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* A family of sections $\{q_\tau\}$, $\tau \in (\tau_1, \tau_2)$, determines a prime end e of the domain D if, for every sequence of numbers $\{\tau^{(n)}\}$ $(n = 1, 2, ...)$, strictly monotonically converging to τ_i $(i = 1$ or $i = 2)$, the sequence of sections $\{q_{\tau^{(n)}}\}$ determines the prime end e.

empty for $r \in (\tau_1, \tau_2)$, with $\tau_1 = 0$ if $\tau_2 \neq +\infty$. Suppose that $\{K_r\}$ is a family of sections defining some simple end e of the domain D, where K_r is a component of the set S'_r. Let the function $d(f(K_r))$ be a measurable function of the variable r, and let the set $D_1 = \bigcup K_r$ be measurable. Let the integral $I(f, D, F)$ be finite, where

$$F(x, y, Z) = u^2(x)\,(1 + |y|^2)^{-2}\|Z\|^2,$$

$u(x)$ is a continuous positive function in D such that $u(x) \to 0$ as $x \to e$, $\inf\limits_{x \in g/g_1} u(x) > 0$, where g is some subdomain into which the simple end e enters, and $g_1 \subset g$ is any subdomain into which the simple end e enters.

Then for every segment $[r_1, r_2] \subset (\tau_1, \tau_2)$ there exists an $\bar{r} \in [r_1, r_2]$ such that

$$d_\rho(f(K_{\bar{r}})) \leq [2\pi\varkappa(r_1, r_2) I(f, D_1, F)]^{1/2} \ln^{-1/2} r_2/r_1, \tag{8}$$

where

$$\varkappa(r_1, r_2) = \sup_{[r_1, r_2]} \beta(r) \Big/ \inf_{x \in D_1} u^2(x),$$

$\beta(r) = l(K_r)/\pi r$, and $l(K_r)$ is the length of the arc K_r.

Remark. Inequality (8) makes it possible to draw conclusions about the behavior of the mapping $y = f(x)$ in a neighborhood of the simple end e, when this neighborhood has the form of a "zero angle," whose "vertex" is the body of the simple end e.

Example. Suppose that for the mapping $y = f(x)$ the conditions of Theorem 5 are satisfied; let $u(x) = x_2^\alpha$, $-1 \leq \alpha < 0$, and let the domain D lie in the upper half-plane. Suppose that the simple end e has ∞ as its body and that there exists a subdomain g, into which the simple end e enters, such that g is contained between two parallel straight lines. Then, using inequality (8), one can show that under the mapping $y = f(x)$ the simple end e passes into some simple end e^* of the domain Δ.

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Note: Figure translations are in progress. See original paper for figures.

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