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Abstract

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MATHEMATICS

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ON A CLASS OF NONLINEAR OPERATOR EQUATIONS

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This paper establishes a number of theorems on the existence of solutions of nonlinear operator equations for which an energy inequality is valid. Stationary and nonstationary equations with nonlinearities of this type were studied in works (^{1}-^{3}). In them the existence of generalized solutions of the corresponding problems was established. The use of multiplicative inequalities (see (^{4})) makes it possible to prove the existence of strong, and sometimes arbitrarily smooth, solutions. The abstract theorems are illustrated by examples of boundary-value problems for partial differential equations and, in particular, by problems on the strong bending of thin plates.

1. Let three Banach spaces E_0, E_1, E_2 be given, satisfying the condition $E_0 \subseteq E_1 \subseteq E_2$, where the sign $\subseteq$ denotes embedding (^{5}). We shall say that the norms of the spaces E_0, E_1, E_2 satisfy a multiplicative inequality if

$$\|u\|_{E_1} \leq c\|u\|_{E_0}^{\tau}\|u\|_{E_2}^{1-\tau} \quad (0 < \tau < 1,\ c > 0) \tag{1}$$

for every $u \in E_0$.

Consider two scales of Banach spaces $\{E_\alpha\}, \{F_\alpha\}$ $(\alpha \in [\alpha_0, \beta_0])$ (see (^{5})). We shall consider an operator equation of the form

$$Au + Ku = h \tag{2}$$

in these scales. Here A is a linear operator defined on E_α with range in F_α $(\alpha \in [\alpha_0, \beta_0])$, K is a nonlinear operator, and h is a given element belonging to F_α. By a solution of equation (2) we shall mean an element $u \in E_\alpha$ satisfying the given equation.

Let there exist a Banach space E_0 such that $E_{\alpha_0} \subseteq E_0$. Suppose that for each pair E_0, E_α $(\alpha \in [\alpha_0, \beta_0])$ there exists a finite number of Banach spaces E_{0i}^{α} $(i = 1, 2, \ldots, N_0)$ such that $E_\alpha \subseteq E_{0i}^{\alpha} \subseteq E_0$ $(i = 1, 2, \ldots, N_0)$, and each triple satisfies a multiplicative inequality with exponent $\tau_{\alpha i}$.

Theorem 1. *Suppose the following conditions are fulfilled:*

a) *The operator A is linear and bounded, mapping E_α onto F_α $(\alpha \in [\alpha_0, \beta_0])$, and satisfies the condition*

$$\|Au\|_{F_\alpha} \geq c_{1\alpha} \|u\|_{E_\alpha} \quad (\forall u \in E_\alpha,\ c_{1\alpha} > 0). \tag{3}$$

b) *The nonlinear operator K, mapping E_α into F_α, is completely continuous, and the inequality*

$$\|Ku\|_{F_\alpha} \leq \sum_{k=1}^{N} \prod_{i=1}^{N_0} d_{ki\alpha} \|u\|^{r_{ki\alpha}}_{E^\alpha_{0i}} \|u\|^{s_{ki\alpha}}_{E_0}, \tag{4}$$

holds, where $s_{ki\alpha}$ are arbitrary nonnegative numbers, while $r_{ki\alpha}$ satisfy the condition

$$\sum_{i=0}^{N_0} r_{ki\alpha} \tau_{\alpha i} < 1 \tag{5}$$

for every $k = 1, 2, \dots, N$ and $\alpha \in [\alpha_0, \beta_0]$.

c) The operator $A + K$ is such that for every $u \in E_\alpha$ the relation

$$\|u\|_{E_0} \leq c_{2\alpha} \|(A + \lambda K)u\|_{F_\alpha} \quad (\alpha \in [\alpha_0, \beta_0],\ \lambda \in [0, 1]). \tag{6}$$

holds.

Then for any function $h \in F_\alpha$ $(\alpha \in [\alpha_0, \beta_0])$ there exists at least one solution of equation (2) belonging to E_α.

The proof is carried out with the aid of the Leray–Schauder principle $(^6)$ and multiplicative inequalities of type $(^1)$.

Let the space E_0 be embedded in a Hilbert space, and suppose that for arbitrary $\varphi \in F_\alpha$, $u \in E_\alpha$ $(\alpha \in [\alpha_0, \beta_0])$ one has

$$|(\varphi, u)_H| \leq c_{3\alpha} \|\varphi\|_{F_\alpha} \|u\|_{E_0}, \quad (c_{3\alpha} > 0). \tag{7}$$

Theorem 2. Suppose that conditions a), b) of Theorem 1 are satisfied. The operator $A + K$ is such that

$$((A + \lambda K)u, u)_H \geq c_{4\alpha} \|u\|^2_{E_0}, \quad (u \in E_\alpha,\ c_{4\alpha} > 0). \tag{8}$$

Then for any element $h \in F_\alpha$ there exists at least one solution of equation (2) belonging to E_α.

2. Consider the problem

$$Lu + Pu = f(x) \text{ in } \Omega, \qquad B_j u = 0 \text{ on } \Gamma \quad (j = 1, 2, \dots, m), \tag{9}$$

where L is a properly elliptic operator of order $2m$; the boundary conditions B_j cover the operator L (see (7)); the order of the boundary operators does not exceed $2m - 1$; $Pu = P(u, Du, \dots, D^{2m-1}u)$.

Assume that the linear problem is uniquely solvable for any function $f(x) \in L_p(\Omega)$. The totality of functions $u(x) \in W_p^{2m}(\Omega)$ satisfying the boundary conditions forms a closed subspace in $W_p^{2m}(\Omega)$, which we denote by $W_p^{2m}(\Omega, B)$, where $W_p^{2m}(\Omega)$ is the Sobolev space (8); the boundary Γ of the domain Ω is sufficiently smooth.

Theorem 3. Suppose that in a two-dimensional domain Ω the following conditions are satisfied:

α) The operator P is completely continuous as an operator acting from $W_p^{2m}(\Omega, B)$ into $L_p(\Omega)$, and

$$\|Pu\|_{L_p} \leq \sum_{k=1}^{N} d_k \left\| \prod_{j=0}^{2m-1} |D^j u|^{s_{jk}} \right\|_{L_p}, \tag{10}$$

where s_{jk} are arbitrary nonnegative numbers for $j = 0, 1, \dots, m-1$ and any k, while the remaining exponents are nonnegative and satisfy the condition: for $s_{mk} > 1/2p$ and any k

$$\sum_{j=m+1}^{2m-2j} \left(\frac{j}{2m} + \frac{1}{pm} \right) s_{jk} + \left(s_{mk} - \frac{1}{2p} \right) \left(\frac{1}{2} + \frac{1}{pm} \right) + s_{2m-1,k} \left(\frac{2m-1}{2m} + \frac{2s_{2m-1,k} - 1}{2mp\, s_{2m-1,k}} \right) < 1,$$

for $s_{mk} \leq 1/2p$ and any k

$$\sum_{j=m+1}^{2m-2} \left(\frac{j}{2m} + \frac{1}{pm} \right) s_{jk} + s_{2m-1,k} \left(\frac{2m-1}{2m} + \frac{2s_{2m-1,k} - 1}{2mp\, s_{2m-1,k}} \right) < 1.$$

β) The operator $L + P$ is such that

$$((Lu + \lambda Pu), u)_{L_2} \geq c_5 \|u\|^2_{W_2^m} \quad (u \in W_p^{2m}(\Omega, B),\ \lambda \in [0, 1]).$$

Then for any function $f(x) \in L_p(\Omega)$ $(p > 1)$ there exists at least one solution belonging to $W_p^{2m}(\Omega, B)$.

In the proof we use Theorem 2. We have $E_\alpha = W_p^{2m}(\Omega, B)$, $F_\alpha = L_p(\Omega)$, $E_0 = W_2^m(\Omega)$ $(\alpha = 1/p)$, $E_{0i}^\alpha = C^i(\Omega \cup \Gamma)$ $(i = m+1, \dots$

$\cdots$, $2m - 2)$, $E_{02m-1}^\alpha = W_{2ps_{2m-1,k}}^{2m-1}(\Omega)$. Using the embedding theorems and Hölder’ s inequality, we obtain, for example, for $s_{mk} > 1/2p$,

$$\left\|\sum_{j=0}^{2m-1} |D^j u|^{s_{jk}}\right\|_{L_p} \leq \sum_{j=m+1}^{2m-2} d_{kj} \|u\|_{W_2^m}^{r_k} \|u\|_{C^j}^{s_{jk}} \|u\|_{C^m}^{(s_{mk}-1)p} \|u\|_{W_{2ps_{2m-1,k}}^{2m-1}}^{s_{2m-1,k}},$$

where

$$r_k = \sum_{j=0}^{m-1} s_{jk} + 1/2p, \quad k = 1, 2, \dots, N;$$

hence, by virtue of the multiplicative inequalities (4) and the assumptions of this theorem, we obtain the proof.

Let the operator Pu have the form

$$|Pu| \leq \sum_{k=m}^{2m-1} \sum_{|\delta|=k} \left|V_\delta(u, Du, \dots, D^{m-1}u)\right| \, |D^\delta u|^{s_\delta}. \tag{11}$$

Theorem 4. Suppose that condition β) of Theorem 3 is satisfied,

$$\left|V_\delta(u, Du, \dots, D^{m-1}u)\right| \leq \prod_{j=0}^{m-1} d_{j\delta} |D^j u|^{r_{j\delta}} \quad (r_{j\delta} \geq 0), \tag{12}$$

where the $r_{j\delta}$ are arbitrary, and $s_\delta \geq 0$ satisfy the inequalities:

for $s_m > 1/p$

$$\left(s_m - \frac{1}{p}\right)\left(\frac{1}{2} + \frac{1}{pm}\right) < 1, \quad s_\delta\left(\frac{\delta}{2m} + \frac{s_\delta - 1}{mps_\delta}\right) < 1, \quad \delta = m+1, \dots, 2m-1,$$

for $s_m \leq 1/p$

$$s_\delta\left(\frac{\delta}{2m} + \frac{s_\delta - 1}{mps_\delta}\right) < 1, \quad \delta = m+1, \dots, 2m-1.$$

Then for every $f(x) \in L_p(\Omega)$ $(p > 1)$ there exists at least one solution belonging to $W_p^{2m}(\Omega, B)$.

Consider the three-dimensional case, when Pu is given by (11). For simplicity we assume that $m \geq 2$; then we have

Theorem 5. Suppose that condition β) of Theorem 3 is satisfied and inequality (12) holds. Then, if the following inequalities are satisfied: $r_{j\delta}$ arbitrary for $j = 0, 1, \dots, m-2$ and for any δ, if $r_{m-1\delta} > 1/p$, then for $\delta = m, \dots, 2m-1$

$$\left(r_{m-1\delta} - \frac{1}{p}\right)\left(\frac{2mp-p}{4mp+p-6}\right) + s_\delta\left(\frac{\delta}{2m} + \frac{3}{2mp}\left(1 - \frac{5}{6s_\delta}\right)\right) < 1,$$

if $r_{m-1\delta} \leq 1/p$, then for $\delta = m, \dots, 2m-1$

$$s_\delta\left(\frac{\delta}{2m} + \frac{3}{2mp}\left(1 - \frac{5}{6s_\delta}\right)\right) < 1,$$

then for every function $f(x) \in L_p(\Omega)$ $(p > 1)$ there exists at least one solution of problem (9), belonging to the space $W_p^{2m}(\Omega, B)$.

For $m = 1$ the conditions on $r_{j\delta}$ and s_δ are more cumbersome than in the preceding theorem; in particular, if $r_{0\delta} = 1$, $s_\delta = 1$, then for every function $f(x) \in L_p(\Omega)$ $(p \geq 6/5)$ there exists at least one solution of problem (9), belonging to $W_p^{2m}(\Omega, B)$.

3. Let us consider an application of Theorem 1 to problems of the theory of thin plates. The basic equations in the case of variable thickness and rigidity are described by the system (9):

$$A_D\omega = L(\omega, F) + q, \qquad A_{1/h}F = \frac{1}{2}EL(\omega, \omega),$$

where

$$L(F, \omega) = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 \omega}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 \omega}{\partial x \partial y} + \frac{\partial^2 \omega}{\partial x^2}\frac{\partial^2 F}{\partial y^2}. \tag{13}$$

$$\begin{aligned} A_m u = {} & \frac{\partial^2}{\partial x^2}\left(m\frac{\partial^2 u}{\partial x^2}\right) + 2\frac{\partial^2}{\partial x\,\partial y}\left(m\frac{\partial^2 u}{\partial x\,\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(m\frac{\partial^2 u}{\partial y^2}\right) + \\ & + \mu\left[\frac{\partial^2}{\partial y^2}\left(m\frac{\partial^2 u}{\partial x^2}\right) - 2\frac{\partial^2}{\partial x\,\partial y}\left(m\frac{\partial^2}{\partial x\,\partial y}\right) + \frac{\partial^2}{\partial x^2}\left(m\frac{\partial^2 u}{\partial y^2}\right)\right]. \end{aligned}$$

Depending on the nature of the fastening of the edge of the plate, we have the boundary conditions:

$$F\big|_\Gamma = 0, \qquad \left.\frac{\partial F}{\partial \nu}\right|_\Gamma = 0, \qquad \omega\big|_\Gamma = 0, \qquad \left.\frac{\partial \omega}{\partial \nu}\right|_\Gamma = 0, \tag{14}$$

$$F\big|_\Gamma = \left.\frac{\partial F}{\partial \nu}\right|_\Gamma = 0, \qquad \omega\big|_\Gamma = \left.\left(\Delta\omega - \frac{1-\mu}{\rho}\frac{\partial \omega}{\partial \nu}\right)\right|_\Gamma = 0, \tag{15}$$

$$F\big|_\Gamma = \frac{\partial F}{\partial \nu}\bigg|_\Gamma = 0, \qquad \omega\big|_\Gamma = \left[\Delta\omega - \left(\frac{1-\mu}{\rho} - k\right)\frac{\partial \omega}{\partial \nu}\right]\bigg|_\Gamma = 0, \tag{16}$$

$k > 0$, ν is the outward normal; for the boundary conditions (15) and (16) we assume that the stiffness on the boundary is constant and minimal.

Theorem 6. Let $q(x,y) \in W_p^r(\Omega)$, $D(x,y), h(x,y) \in C^{2+r}(\Omega \cup \Gamma)$; then there exists at least one solution of each stated problem, belonging to the space $W_p^{4+r}(\Omega) \times W_p^{4+r}(\Omega)$.

Theorem 6 is a generalization of the results of the works (^{10,11}). An analogous theorem is valid for anisotropic plates under the boundary condition (14).

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Note: Figure translations are in progress. See original paper for figures.