NONSTANDARD ANALYSIS AND HOMEOMORPHISM OF \(B\)-SPACES

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Abstract

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NONSTANDARD ANALYSIS AND HOMEOMORPHISM OF B-SPACES

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In 1966 M. I. Kadets, in paper ([1]), showed that all infinite-dimensional separable Banach spaces are homeomorphic. This gave an answer to an important problem of Fréchet and Banach. The proof proposed by him is based on the method of coordinates and the method of decomposition ([2]).

It would be natural to approach this problem and questions on the homeomorphism of metric spaces from more general positions. In the present article some applications of model theory and nonstandard analysis to concrete problems of topological equivalence of Banach spaces and properties of nonstandard B-spaces are considered. The article gives a new proof of Kadets' theorem and considers the topological equivalence of nonstandard B-spaces. The basic definitions and theorems used in the present article are set forth in ([3–5]).

Let φ be a formula of the narrow predicate calculus containing no predicate symbols other than $=$, $<$, $\|\cdot\|$; let F be the class of all such formulas.

Let $^*R$ be a nonstandard model of analysis ([5]). The subset $[0,1]$ of the model R (R is the set of real numbers) generates in $^*R$ the standard interval $^*[0,1]$ (see ([3]), p. 325).

Let $f:[0,1]\to R$ be a function. Consider a binary predicate $\sigma_f(x,y)$ such that $\sigma_f(x,y)$ if and only if $x\in[0,1]$ and $y=f(x)$. In R the axioms

$$(1)\quad (\forall x)(\forall y)\big(\Phi_{(0,0)}(\sigma_f,x,y)\to\Phi_{(0)}(t_{0,1},x)\big).$$

$$(2)\quad (\forall x)\big(\Phi_{(0)}(t_{0,1},x)\to(\exists y)(\forall z)(\Phi_{(0,0)}(\sigma_f,x,y)\ \&\ \Phi_{(0,0)}(\sigma_f,x,z)\to\to\Phi_{(0,0)}(e,y,z))\big).$$

are satisfied.

The definition of Φ_τ is given in ([5]).

Axioms (1), (2) are formulated in terms of the language Λ ([5]) and, therefore, are true in $^*R$. Then we put $f^*:{}^*[0,1]\to{}^*R$, if

$$f^*(r) = z \leftrightarrow \sigma_f(r, z) \quad {}^*R.$$

Denote by ${}^*C[0,1]$ the set of all internal functions (see ([5]), p. 42) $q : {}^*[0,1] \to {}^*R$ that are continuous (in the classical sense) ([5]). We identify the function $f \in C[0,1]$ with the function $f^* \in {}^*C[0,1]$; hence, $C[0,1] \subset {}^*C[0,1]$. We introduce a norm on ${}^*C[0,1]$ as follows:

$$\|q\| = \max_{t \in {}^*[0,1]} |q(t)|.$$

According to ([5]), ${}^*C[0,1]$ is a nonstandard B-space.

The construction of nonstandard B-spaces may be carried out by the method of ultrapowers (see ([6]), p. 297). Let ${}^*R = R^\omega/D$ be a countable ultrapower of R ([3]). Here and below D is a nonprincipal ultrafilter. Consider $C[0,1]^\omega/D$, the countable ultrapower of $C[0,1]$. On $C[0,1]^\omega/D$ we introduce the norm $\|\cdot\|_D$ in the following way. If $q/D \in {}^*R$, then $\|f/D\|_D = q/D$ if and only if $\{n : \|f(n)\| = q(n)\} \in D$.

Proposition 1. ${}^*C[0,1]$ is isometric to $C[0,1]^\omega/D$.

Proof. Represent the structure ${}^*[0,1]$ in the form of a set of certain internal predicates over ${}^*R = R^\omega/D$. Consider a function $f : {}^*[0,1] \to R$.

According to (5), the function f is determined by a two-place predicate S_f in ${}^*R$ in such a way that S_f is an internal predicate if and only if f is an internal function. Moreover, f is continuous if and only if a certain axiom δ_1 of the language Λ is satisfied in ${}^*R$. Then ${}^*C[0,1]$ is the totality of all internal predicates S_f in ${}^*R = R^\omega/D$ satisfying the conditions δ_1, and

$$\delta_2 = (\forall x)(\Phi_{(0)}(t_{0,1}, x) \to (\exists y)\Phi_{(0,0)}(S_f, x, y)) \;\&$$

$$\& \;(\forall x)(\forall y)(\forall z)(\Phi_{(0,0)}(S_f, x, y) \;\&\; \Phi_{(0,0)}(S_f, x, z) \to \Phi_{(0,0)}(e, y, z)).$$

Here: q is the order relation $<$; e is the relation $=$. Every function $q \in C[0,1]$ can be identified with a predicate S_q in R satisfying the conditions $\delta_1 - \delta_2$. Then every element $\lambda/D \in C[0,1]^\omega/D$ can be regarded as $\langle S_{\lambda(1)}, \dots, S_{\lambda(n)}, \dots\rangle/D$, since $\lambda(n) \in C[0,1]$.

At the same time, every internal predicate S in ${}^*R = R^\omega/D$ satisfying $\delta_1 - \delta_2$ has the form $\langle S_{\lambda(1)}, \dots, S_{\lambda(n)}, \dots\rangle/D$ for some element $\lambda/D \in C[0,1]^\omega/D$. Consequently, $C[0,1]^\omega/D$ is represented as the totality of all internal predicates in ${}^*R = R^\omega/D$ satisfying the conditions δ_1 and δ_2. The linearity of the correspondence between ${}^*C[0,1]$ and $C[0,1]^\omega/D$ is obvious, since in the ultrapower addition of elements and multiplication by a scalar are performed componentwise (see (3)).

The isometry between $^*C[0,1]$ and $C[0,1]^\omega/D$ is established by the definition of the norms $\|\cdot\|$ in $C^*[0,1]$ and $\|\cdot\|_D$ in $C[0,1]^\omega/D$, using the nonstandard variant of Weierstrass' theorem (5). We note that an analogue of Proposition 1 also holds for other B-spaces.

By the symbol $\mathfrak{A} \sim \mathfrak{N}$ we shall denote a homeomorphism (i.e. a one-to-one and bicontinuous mapping) of $\mathfrak{A}$ onto $\mathfrak{N}$ (1).

Proposition 2. $^*l_2 \sim {}^*C[0,1]$ (for the definition of $^*l_2$, see (5)).

The homeomorphism between $^*C[0,1]$ and $^*l_2$ is the extension of the homeomorphism between l_2 and $C[0,1]$ (2).

Theorem 1. Let $\mathfrak{N}$ be a separable infinite-dimensional Banach space; then $^*\mathfrak{N}$, which is a nonstandard model of the B-space $\mathfrak{N}$, is homeomorphic to $C[0,1]$ (and hence also homeomorphic to l_2).

In proving the theorem we use the method of finite approximation proposed by Keisler. The basic definitions and theorems relating to this method are given in (7, 8).

Proof. Identify $\mathfrak{N}$ with that subspace of $C[0,1]$ which is isometric to it; $^*\mathfrak{N}$ is the totality of those internal functions from $^*C[0,1]$ which are extensions of the corresponding standard functions from $\mathfrak{N}$ (see (5), Theorem 2.11.4).

$$\overline{\overline{\mathfrak{N}}} = \overline{\overline{C[0,1]}} = {}^*\overline{\overline{R}} = 2^{\aleph_0}.$$

Let $\{b_\beta : \beta < 2^\omega\} = \mathfrak{N}$ be an enumeration of all elements of $^*\mathfrak{N}$, and let $a \in {}^*C[0,1]^{2^\omega \times 2^\omega}$ be a function mapping $2^\omega \times 2^\omega$ into $C[0,1]$ such that $\{a_{1\beta} : \beta \geq 2^\omega\}$ are all elements of $^*C[0,1]$. Here $a_{\alpha\beta}$ denotes $a(\alpha\beta)$; $\alpha, \beta < 2^\omega$. In $\mathfrak{N}$ the following axiom is satisfied:

$$\varphi = (\exists v_{01} \forall v_{11} \exists v_{02} \forall v_{12} \ldots \exists v_{0\beta} \forall v_{1\beta})_{\beta < 2^\omega} \ \& \ \Gamma,$$

where Γ is the totality of formulas of the form:

$$(3) \quad \&\{v_{0\delta} \neq v_{0\gamma} : \delta \neq \gamma,\ \delta, \gamma < 2^\omega\};\ \vee\{v_{1\beta} = v_{0\gamma} : \gamma < 2^\omega\},$$

$$(4) \quad \&\{\&\{\vee\{\&\{\|v_{0\beta} - v_{0\gamma}\| < \varepsilon : \|b_{0\beta} - b_{0\gamma}\| < \delta;\ \gamma < 2^\omega\}\delta > 0\}\beta < 2^\omega\}\varepsilon > 0\}.$$

$$\neg\{\vee\{\vee\{\&\{\vee\{\|v_{0\gamma} - v_{0\beta}\| < \delta : \|b_{0\gamma} - b_{0\beta}\| < \varepsilon;\ \gamma < 2^\omega\}\delta > 0\}\beta < 2^\omega\}\varepsilon > 0\}\}. \quad (5)$$

Let us represent φ as follows: $\varphi(Qx)\psi$. Then $\mathfrak{N} \vDash \psi[b]$. Such a formula φ will be an axiom of the language L' (see ([8]), if $m = 2^{\aleph_0}$.

In ${}^*\mathfrak{N}$ the axiom φ is satisfied, since the identity mapping of ${}^*\mathfrak{N}$ onto itself will be a homeomorphism.

From Corollary 5.5.5 ([9]) it follows that ${}^*C[0,1]$ [isometric $C[0,1]^\omega/D$] is an ω_1-saturated model, if ${}^*R = R^\omega/D$ and the ultrafilter D is countably incomplete (according to Theorem 2.11.4 ([5]), ${}^*\mathfrak{N} = \mathfrak{N}^\omega/D$ is ω_1-saturated). The axiom φ is prenex in F and, consequently, admissible ([8]). If $\neg\varphi^*$ is false in ${}^*C[0,1]$, then for some $\theta \in A(\varphi)$ ($A(\varphi)$ is the set of finite approximations of φ) $\theta \dashv {}^*\mathfrak{N}$, but $\neg\theta \dashv {}^*C[0,1]$.

By Lemma 1.4 ([8]), θ is formulated in terms of $=$, $\|\cdot\|$, $<$, i.e. is an F-axiom. Then, by Lemma 3.1 ([7]), $\neg\varphi^* \dashv {}^*C[0,1]$; hence, according to ([7]), there is no winning strategy for $(Q^*, x), \neg\psi$.

Then the set $\{h_{1\beta} : \beta < 2^\omega\}$ of functions $h_{1\beta}$ having constant values $a_{1\beta}$ is not a winning strategy for $(Q^*x), \neg\psi$. Consequently, there exists a function $c \in {}^*C[0,1]^{2^\omega \times 2^\omega}$ such that $c_{1\beta} = a_{1\beta}$, $\beta < 2^\omega$, and ${}^*C[0,1] \vDash \psi[c]$.

From (3), (4), and (5) it follows that the correspondence $f : b_{0\beta} \mapsto c_{0\beta}$ realizes the required homeomorphism. From Theorem 1, Proposition 2 follows immediately in the case $\mathfrak{N} = l_2$.

Theorem 2. If ${}^*l_2 \sim {}^*\mathfrak{N}$, then $l_2 \sim \mathfrak{N}$, where $\mathfrak{N}$ is a space with a basis.

In the proof we use the fact that all finite-dimensional B-spaces of a given dimension n are isomorphic; this result generalizes to the case of nonstandard B-spaces (cf. ([5])).

The Schauder basis $\{e_n\}_{n\in N}$ in $\mathfrak{N}$ generates in ${}^*\mathfrak{N}$ a basis $\{{}^*e_n\}_{n\in {}^*N}$ such that ${}^*e_i = e_i$ for all finite $i \in N$ (see ([6])). Further, acting by methods of nonstandard analysis similar to those set forth in ([5],[6],[3]), we prove the existence of a one-to-one and mutually continuous mapping between the unit spheres of l_2 and $\mathfrak{N}$, which extends to a homeomorphism of l_2 onto $\mathfrak{N}$. The principal result of the present paper is

Theorem 3. If $\mathfrak{N}$ is a separable infinite-dimensional B-space, then $\mathfrak{N} \sim l_2$.

Proof. By Theorem 1, ${}^*C[0,1] \sim {}^*\mathfrak{N}$, but, by Proposition 2, $l_2 \sim {}^*C[0,1] \sim {}^*\mathfrak{N}$; hence ${}^*l_2 \sim {}^*\mathfrak{N}$, whence, by Theorem 2, $l_2 \sim \mathfrak{N}$, if $\mathfrak{N}$ is a space with a basis.

Let now $\mathfrak{N}$ be an arbitrary infinite-dimensional separable B-space. In $\mathfrak{N}$ there is a subspace $\mathfrak{M}$ with a basis. Then, by Theorem 8.1 ([2]), $\mathfrak{M} \mid \mathfrak{N}$, but $\mathfrak{M} \sim l_2$; consequently, $l_2 \mid \mathfrak{N}$, and, by Theorem 8.2 ([2]), $\mathfrak{N} \sim l_2$.

Let $\mathfrak{A}$ be a B-space, and let $\mathfrak{A}^I_D \mid G$ (see ([11])) be a bounded ultrapower of the system $\mathfrak{A}$. Then $\mathfrak{A}^I_D \mid G$ will be a nonstandard B-space (cf. (6)). Moreover, with the help of ([11]) one proves

Remark 1. A model $\mathfrak{N}$ will be a nonstandard model of the B-space $\mathfrak{A}$ if and only if $\mathfrak{N}$ is isometric to the bounded ultrapower $\mathfrak{A}^I_D \mid G$. The analogue of Remark 1 is also valid for other algebraic systems.

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References

1. M. I. Kadec, *Functional Analysis*, 1, No. 1, 61 (1967).
2. C. Bessaga, *Fund. Math.*, 56, 251 (1963).
3. A. Robinson, *Introduction to Model Theory and to the Metamathematics of Algebra*, "Nauka," 1967.
4. A. I. Mal' tsev, *Proceedings of the Fourth All-Union Mathematical Congress*, 1, Leningrad, 1963.
5. A. Robinson, *Non-Standard Analysis*, Amsterdam, 1966.
6. A. Robinson, *The Theory of Models*, Amsterdam, 1965.
7. H. J. Keisler, *The Theory of Models*, Amsterdam, 1965.
8. H. J. Keisler, *J. Symb. Logic*, 30, 3 (1965).
9. Chang Keisler, *Continuous Model Theory*, Princeton, 1966.
10. L. A. Lyusternik, V. I. Sobolev, *Elements of Functional Analysis*, "Nauka," 1965.
11. H. J. Keisler, *Trans. Am. Math. Soc.*, 107, 285 (1963).

Note: Figure translations are in progress. See original paper for figures.