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**Abstract**

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# GENERALIZED STEENROD HOMOLOGY GROUPS

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Let $L$ and $L'$ be some simplicial complexes. A mapping $f : L \to L'$ will be called a closed simplicial embedding if it is a simplicial mapping that maps $L$ homeomorphically onto some closed subcomplex of the complex $L'$.

We shall consider a certain category $\mathfrak{A}$, whose objects are locally finite simplicial complexes of a specified kind, and whose mappings are closed simplicial embeddings of a specified kind. In particular, for every complex $L$ that is an object of the category $\mathfrak{A}$, the mapping $1_L$, which is the identity mapping of the complex $L$ onto itself, is defined in this category. In what follows we shall assume that the category $\mathfrak{A}$ has the following properties:

$1^\circ$. If $L_1$ and $L_2$ are arbitrary objects of the category $\mathfrak{A}$, then the union $L_1 \cup L_2$ (without identifications) is also an object of the category $\mathfrak{A}$, and the natural mappings $L_1 \to L_1 \cup L_2$ and $L_2 \to L_1 \cup L_2$ belong to the category $\mathfrak{A}$. More precisely, this means the following: for any objects $L_1, L_2 \in \mathfrak{A}$ there are in the category $\mathfrak{A}$ an object $L$ and mappings $f_1 : L_1 \to L$, $f_2 : L_2 \to L$ such that $f_1(L_1) \cap f_2(L_2) = \emptyset$ and $f_1(L_1) \cup f_2(L_2) = L$ (recall that every mapping of the category $\mathfrak{A}$ is a closed simplicial embedding and, in particular, a homeomorphism).

$2^\circ$. If $f : L \to L_1$ and $g : L \to L_2$ are mappings belonging to the category $\mathfrak{A}$, then the complex $L^*$, composed of $L_1$ and $L_2$, having common part $f(L) = g(L)$, also belongs to the category $\mathfrak{A}$, and the natural mappings $L_1 \to L^*$, $L_1 \to L^*$ belong to the category $\mathfrak{A}$. More precisely, this means the following:

If $f : L \to L_1$ and $g : L \to L_2$ are mappings belonging to the category $\mathfrak{A}$, then there exists in the category $\mathfrak{A}$ a complex $L^*$ and mappings $\alpha : L \to L^*$, $\beta : L_1 \to L^*$, $\gamma : L_2 \to L^*$, for which $\beta \circ f = \gamma \circ g = \alpha$, $\beta(L_1) \cup \gamma(L_2) = L^*$, and, moreover, for zero-dimensional skeleta the equality $\beta(L_1^0) \cap \gamma(L_2^0) = \alpha(L^0)$ holds. In other words, the complex $L^*$ is the union of its two subcomplexes $\beta(L_1)$ and $\gamma(L_2)$, the intersection of which contains the subcomplex $\alpha(L)$ (although it may not coincide with it), while the subcomplexes $\beta(L_1)$ and $\gamma(L_2)$ have no common vertices except those that belong to the subcomplex $\alpha(L)$.

Suppose further that some group $G$ is fixed and that, in each complex $L$ which is an object of the category $\mathfrak{A}$, a certain class of chains $\alpha(L)$ over the coefficient group $G$ is specified and a differential operator $d$ is defined, satisfying the condition $d \circ d = 0$, and that the following conditions are fulfilled (the lower indices indicate a certain filtration, hereafter called dimension):

1. If $x_p \in \alpha(L)$, then also $dx_p \in \alpha(L)$.
2. If $x_p, y_p \in \alpha(L)$, then $x_p + y_p \in \alpha(L)$.
3. If $x_p \in \alpha(L)$ and $g$ is some endomorphism of the group $G$, then $gx_p \in \alpha(L)$.
4. If $x_p \in \alpha(L)$ and $f : L \to L_1$ is a mapping of the category $\mathfrak{A}$, then $f(x_p) \in \alpha(L_1)$.

We note that if $K$ is some ring and $G$ is the additive group of this ring, then the correspondence $g \mapsto kg$ for any $k \in K$ defines an endomorphism of the group $G$. Thus, in this case it follows from axiom 3, in particular, that if $x_p \in a(L)$, then also $kx_p \in a(L)$ for any $k \in K$. Moreover, from axiom 3 it follows (in the case of any abelian group $G$) that if $x_p \in a(L)$, then $-x_p \in a(L)$. Together with axiom 2 this means that all $p$-dimensional chains belonging to $a(L)$ form an abelian group $a_p(L)$ (with respect to the usual addition of chains). Thus the mappings $d : a_p(L) \to a_{p-1}(L)$ and $f : a_p(L) \to a_p(L_1)$ (for any mapping $f : L \to L_1$ belonging to the category $\mathfrak{A}$) are homomorphisms.

Finally, let $L$ be an arbitrary object of the category $\mathfrak{A}$, and let $\varphi$ be some mapping of its zero-dimensional skeleton $L^0$ into a compact metric space $\Phi$. We shall call the mapping $\varphi$ regular (cf. ([1])) if, for every $\varepsilon > 0$, only a finite number of simplices of the complex $L$ have their vertices mapped into a subset of the compactum $\Phi$ whose diameter is less than $\varepsilon$.

If $L \in \mathfrak{A}$, $x_p \in a_p(L)$, and $\varphi : L^0 \to \Phi$ is a regular mapping, then the triple $(L, \varphi, x_p)$ will be called a $p$-dimensional regular $(\mathfrak{A}, a)$-chain of the space $\Phi$ over the coefficient group $G$. If $dx_p = 0$, then the triple $(L, \varphi, x_p)$ will be called a regular $p$-dimensional $(\mathfrak{A}, a)$-cycle of the space $\Phi$ over the group $G$.

Two $p$-dimensional regular $(\mathfrak{A}, a)$-cycles $(L_1, \varphi_1, x'_p)$ and $(L_2, \varphi_2, x''_p)$ of the space $\Phi$ over the group $G$ are called homologous to each other if there exist mappings $f_1 : L_1 \to L$, $f_2 : L_2 \to L$ of the category $\mathfrak{A}$ and a $(p+1)$-dimensional regular $(\mathfrak{A}, a)$-chain $(L, \varphi, x_{p+1})$ such that $\varphi$ coincides with $\varphi_1$ on $f_1(L_1)$ (i.e. $\varphi \circ f_1 = \varphi_1$) and with $\varphi_2$ on $f_2(L_2)$ (i.e. $\varphi \circ f_2 = \varphi_2$), and $dx_{p+1} = f_1(x'_p) - f_2(x''_p)$.

It follows from this definition in a trivial way that if $(L, \varphi, x_p)$ is some $p$-dimensional regular $(\mathfrak{A}, a)$-cycle of the space $\Phi$ over the group $G$, and $f : L \to L_1$ is a mapping of the category $\mathfrak{A}$, and there exists a regular mapping $\varphi_1 : L_1^0 \to \Phi$ satisfying the condition $\varphi_1 \circ (f|L^0) = \varphi$, then the $p$-dimensional regular cycles $(L, \varphi, x_p)$ and $(L_1, \varphi_1, f(x_p))$ are homologous to each other.

Next, it is easy to show that the homology relation is reflexive, symmetric, and transitive (cf. ([1])). Thus the $p$-dimensional regular $(\mathfrak{A}, a)$-cycles $(L, \varphi, x_p)$, $x_p \in a(L)$, of the space $\Phi$ over the group $G$ split into disjoint classes of mutually homologous regular $(\mathfrak{A}, a)$-cycles. These classes are called homology classes of regular $p$-dimensional $(\mathfrak{A}, a)$-cycles of the space $\Phi$ over $G$.

The sum of two $p$-dimensional regular $(\mathfrak{A}, a)$-chains $(L_1, \varphi_1, x'_p)$ and $(L_2, \varphi_2, x''_p)$ is called the $p$-dimensional regular $(\mathfrak{A}, a)$-chain $(L_1 \cup L_2, \varphi, x'_p + x''_p)$, where

$L_1 \cup L_2$ is the union of the complexes $L_1$ and $L_2$ without identifications, and the mapping $\varphi$ coincides with $\varphi_1$ on $L_1$ and with $\varphi_2$ on $L_2$. More precisely, this definition can be formulated as follows. Let $(L_1, \varphi_1, x'_p)$ and $(L_2, \varphi_2, x''_p)$ be two $p$-dimensional regular $(\mathfrak{A}, a)$-chains. Let, further, $L$ be an object of the category $\mathfrak{A}$, and let $f_1 : L_1 \to L$, $f_2 : L_2 \to L$ be mappings of the category $\mathfrak{A}$ satisfying the conditions $f_1(L_1) \cup f_2(L_2) = L$, $f_1(L_1) \cap f_2(L_2) = \emptyset$. Denote by $\varphi : L \to \Phi$ the mapping satisfying the relations $\varphi \circ f_1 = \varphi_1$, $\varphi \circ f_2 = \varphi_2$ (under these conditions the mapping $\varphi$, evidently, is defined and moreover uniquely). Next, put $x_p = f_1(x'_p) + f_2(x''_p)$. We then obtain the $p$-dimensional regular $(\mathfrak{A}, a)$-chain

$$(L, \varphi, x_p) = (L, \varphi, f_1(x'_p) + f_2(x''_p)),$$

which is called the sum of the two $(\mathfrak{A}, a)$-chains under consideration. (By this definition the sum is specified uniquely up to equivalence.)

The sum of two homology classes of regular $p$-dimensional $(\mathfrak{A}, a)$-cycles is defined uniquely as the homology class containing the sum of two $p$-dimensional regular $(\mathfrak{A}, a)$-cycles respectively belonging to the summands.

classes under consideration. The addition of homology classes thus defined turns the totality of all homology classes of $p$-dimensional regular $(\mathfrak{A}, a)$-cycles into a commutative group, which we shall denote by $H_p^{(\mathfrak{A},a)}(\Phi, G)$.

The group $H_p^{(\mathfrak{A},a)}(\Phi, G)$ will be called, by definition, the $p$-dimensional **generalized homological Steenrod group** of the space $\Phi$ over the coefficient group $G$, or, otherwise, the $p$-dimensional $(\mathfrak{A}, a)$-homology group of the space $\Phi$ over the coefficient group $G$.

It is easy to see that if for $\mathfrak{A}$ one takes the category of all locally finite complexes and all their closed simplicial embeddings, and for $a(L)$ one takes the totality of all (infinite) chains of the complex $L$ over the coefficient group $G$ (with the usual understanding of dimension), then in this case the $(\mathfrak{A}, a)$-homology group coincides with the Steenrod group defined in ([1]). In other words, the Steenrod groups can be obtained as a special case of the construction described above. It is precisely this fact that is the reason why the $(\mathfrak{A}, a)$-homology groups defined above have been called generalized Steenrod homological groups.

Moreover, as we shall show in the following note, many of the previously defined homology groups (see ([2–4])) are obtained as special cases of the construction described above. For example, as special cases of this construction one can obtain the $K$-homology groups defined by the author earlier in ([3]), and hence also Sitnikov' s homological groups $\Delta_p(\Phi, G)$, defined by him in ([4]). Thus, the homologies considered by K. A. Sitnikov do not fall outside the circle of Steenrod' s ideas.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*