BASES OF EIGENVECTORS, THE CHARACTERISTIC FUNCTION, AND INTERPOLATION PROBLEMS IN THE HARDY SPACE \(H^2\)

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 513.88+517.432

MATHEMATICS

N. K. NIKOLSKII, B. S. PAVLOV

BASES OF EIGENVECTORS, THE CHARACTERISTIC FUNCTION, AND INTERPOLATION PROBLEMS IN THE HARDY SPACE H^2

(Presented by Academician V. I. Smirnov on 13 XII 1967)

0.1. Let E be a Hilbert space, $\mathfrak{R}$ the Banach algebra of all (linear continuous) operators from E into E, and $\mathfrak{S}_1$ the ideal of nuclear operators in $\mathfrak{R}$. Denote by $H^2(E)$, $H^\infty(\mathfrak{R})$, and $H^1(\mathfrak{S}_1)$ the spaces* of all functions f, regular in the unit disk $D = \{z : \ |z| < 1\}$ of the complex plane, with values respectively in the spaces E, $\mathfrak{R}$, and $\mathfrak{S}_1$, for which (respectively):

$$\|f_2\| = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{it})\|_E^2 dt \right)^{1/2} < \infty,$$

$$\|f\|_\infty = \sup_{z\in D} \|f(z)\|_{\mathfrak{R}} < \infty,$$

$$\|f\|_1 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{it})\|_{\mathfrak{S}_1} dt < \infty.$$

A function Θ, $\Theta \in H^\infty(\mathfrak{R})$, is called inner $(^{1,2})$ if its angular boundary values $\Theta(e^{it})$ are unitary operators in E for almost all values of t, $t \in [0, 2\pi]$.

0.2. We consider the shift operator S in the space $H^2(E)$

$$(Sf)(z) = zf(z), \qquad |z| < 1, \qquad f \in H^2(E),$$

the operator S^* adjoint to S, and its restriction T to some S^*-invariant subspace K, generated by an inner function Θ (see $(^{1,2})$)**

$$T = S^*|K, \qquad K = H^2(E) \ominus \Theta H^2(E). \tag{1}$$

From the work of B. Sz.-Nagy and C. Foiaş (²) it is known that any linear operator A in a Hilbert space for which

$$\|A\| \leq 1, \qquad \lim_{n\to\infty} A^n = \lim_{n\to\infty} A^{*n} = 0, \tag{2}$$

in the sense of strong operator convergence, is unitarily equivalent to an operator T of the form (1), and the function Θ turns out to be the characteristic operator-function (see (²,³)) of the operator A^*. In particular, conditions (2) are satisfied if A is a completely nonunitary contraction that is annihilated by some (scalar) function h, $h \in H^\infty$,

$$h(A) = 0.$$

Such contractions are called (see (²)) contractions of class C_0.

We shall agree to call the operator T basic if the system of its eigenvectors forms a Riesz basis (a basis equivalent to an orthonormal— in the closure of its linear span (see (4)). In the present note* we shall indicate a criterion for the basis property of the operator T in terms of its characteristic function.

In what follows we shall assume that:

I. The part of the spectrum $\sigma(T)$ of the operator T lying in the unit disk D consists of eigenvalues that are simple poles of the resolvent.

II. The system of eigenfunctions of the operator (1) corresponding to eigenvalues lying inside the unit disk D is complete in K.

In terms of the characteristic function Θ, condition I is rewritten as follows: if $z_k \in \sigma(T^*) \cap D$, then

$$\Theta(z) = \Theta_k(z) B_k(z), \qquad |z| < 1, \tag{3}$$

where Θ_k, B_k are inner functions,

$$B_k(z) = \frac{z_k - z}{1 - \overline{z_k} z} \frac{\overline{z_k}}{|z_k|} \pi_k + (I - \pi_k), \qquad |z| < 1,$$

---

* The properties of abstract analogues of the Hardy classes H^p are discussed in detail, for example, in (¹,²). In what follows we use these properties without special reservations.

** Here $\Theta H^2(E) = \{g : \ g(z) = \Theta(z), \ |z| < 1, \ f \in H^2(E)\}$.

π_k is the orthogonal projector in E onto the subspace $\operatorname{Ker}\Theta(z_k)$ of zeros of $\Theta(z_k)$, and $\Theta_k(z_k)$ is a bounded invertible operator in E (see (2)).

Condition II can also be expressed in terms of the characteristic function of the operator T^* (see Lemma 2.1).

0.3. Before proceeding to the statement of the results, let us recall that for operators of the class C_0, property II of completeness of the system of eigen (or root) subspaces is a purely spectral property (see (2)). It follows from our results that the basis property of the operator is no longer such a property (see Theorems 1.2 and 2.4). In this connection we note that the sufficient conditions known up to now for the basis property of a system of eigen (or root) vectors (see (4-7)) essentially used only the spectral characteristics of the operator T.

§ 1. **Bases.** 1.1. Let T be an operator of the form (1), satisfying conditions I and II, and let $\{z_k\}_{k=1}^{\infty} = \sigma(T) \cap D$ be the sequence of its eigenvalues. It is not hard to verify that the system of eigenfunctions of the operator T is formed by the functions**

$$\varphi_k(z) = \frac{\Theta_k^{*-1}(z_k)e}{1-\overline{z_k}z}(1-|z_k|^2)^{1/2}, \qquad e \in \pi_{kE},\ \|e\| = 1. \tag{4}$$

Here π_k is the orthogonal projector onto $\operatorname{Ker}\Theta(z_k)$. The biorthogonal system consisting of eigenfunctions of the operator T^* has the form

$$\psi_k(z) = \Theta_k(z)\frac{e}{1-\overline{z_k}z}(1-|z_k|^2)^{1/2}, \qquad e \in \pi_{kE},\ \|e\| = 1.$$

Relying on the theorems of N. K. Bari (see (4)), and also using some results of D. Sarason' s work (8), one can obtain a set of criteria for the basis property of the operator T in terms of its characteristic function. Namely, the following holds.

1.2. **Theorem.** I. Under the assumptions of 1.1, the following conditions are equivalent:

1) T is a basis operator.

2)

$$\sum_k (1-|z_k|^2)\left\|\pi_k\Theta_k^{-1}(z_k)g(z_k)\pi_k\right\|_{\gamma}^{1} < \infty$$

for every function g,

$$g \in H_0^1(\mathfrak{S}_1) \equiv \{g:\ g \in H^1(\mathfrak{S}_1),\ g(0) = 0\}.$$

II. If, in addition, the system $\{\psi_k\}_{k=1}^{\infty}$ is complete in K, then conditions 1)–2) are equivalent to the following assertions:

3)

$$\sum_k (1-|z_k|^2)\|\pi_k f(z_k)\|_E^2 < \infty$$

for every function $f,\ f \in H_0^2(T) \equiv \{f:\ f \in H^2(E),\ f(0)=0\}$;

$$\sum_k (1-|z_k|^2)\|\pi_k \Theta_k(z_k)^{-1} f(z_k)\|_E^2 < \infty$$

for every function $f,\ f \in H^2(E)$.

4)

$$\sup_k \|\pi_k \Theta_k^{-1}(z_k)\Delta_k\|_{\mathfrak{L}} < \infty^*,$$

$$\sum_{k,n} \frac{(1-|z_k|^2)^{1/2}(1-|z_n|^2)^{1/2}}{1-\overline{z_n} z_k}(e_n, e_k) \le C \sum_k \|e_k\|^2$$

for every sequence $\{e_n\}_1^{\infty},\ e_n \in \pi_n E,\ n \ge 1$, and for every sequence $\{e_n\}_1^{\infty},\ e_n \in \Delta_{nE},\ n \ge 1$.

1.3. Remark. If $T \in C_0$ and conditions 1.1 are satisfied, then $\{\psi_k\}_1^{\infty}$ is a complete system in K (^{1}). Thus part II of Theorem 1.2 gives a criterion for the basis property of the operator T without additional assumptions.

In the proof of Theorem 1.2 we use the following assertions, which are perhaps also of independent interest.

1.4. Lemma. Let $F \in H^{\infty}(\mathfrak{R})$, and let ΓF be the operator in K defined by the equality $(\Gamma F)f = P(Ff)$, where $(Ff)(z) = F(z)f(z),\ |z|<1,\ f \in K$, and P is the orthogonal projection onto K. Then

$$\operatorname{Ker}\Gamma = \{F:\ \Gamma F = 0\} = \Theta H^{\infty}(\mathfrak{R}).$$

1.5. Lemma**. If

$$\mathfrak{F} = \{F:\ F \in H^{\infty}(\mathfrak{R}),\ \Theta^* F \Theta \in H^{\infty}(\mathfrak{R})\}$$

and

$$H^{\infty}(T^*) = \{\Gamma F:\ F \in \mathfrak{F}\},$$

then $H^{\infty}(T^*)$ is isometrically isomorphic to the quotient space

$$\mathfrak{F}/\Theta H^{\infty}(\mathfrak{R}).$$

---

* Part of the results of this note was reported at a seminar on function theory at Leningrad University in May 1967.

** Speaking of the system (4), we mean any of the systems $\{\varphi_{k,i}\}$ obtained from (4) by choosing an orthonormal basis $\{e_k^{(i)}\}$ in π_{kE}.

§ 2. **Interpolation in $H^2(E)$ and bases.** We now consider the interpolation problem

$$\Delta_k f(z_k)(1-|z_k|^2)^{1/2} = w_k, \qquad w_k \in \Delta_{kE}, \quad k \geq 1, \quad f \in H^2(E), \tag{5}$$

and the corresponding restriction operator J

$$Jf = \{\Delta_k f(z_k)(1-|z_k|^2)^{1/2}\}_{k=1}^{\infty}, \qquad f \in H^2(E).$$

In the works (^{9}–^{11}) the case $\dim E = 1$ was studied in detail (in this case always $\Delta_k = I_E$). In particular, it was established there that fulfillment of the collection of conditions

$$\inf_k \left| \prod_{n \neq k} \frac{z_k - z_n}{1 - \overline{z_n} z_k} \frac{z_n}{|z_n|} \right| > 0, \tag{C}$$

$$\sum_k (1-|z_k|^2)|f(z_k)|^2 < \infty, \qquad f \in H^2, \tag{N}$$

is equivalent to the equality $JH^2 = l^2$. In fact, Carleson proved that condition (C) implies (N) (see (^{10})). We shall show that for $\dim E > 1$ problem (5) is solvable in the class $\{w_k\}_1^\infty \in l^2(\Delta_k)$ (see the definition below), if certain conditions analogous to (C) and (N) are satisfied. However, in the general case one can no longer restrict oneself to only one condition of type (C).

The following two lemmas give necessary and sufficient conditions for completeness in K and uniform minimality*** of the system $\{\psi_k\}_1^\infty$ in terms of the function Θ.

$*$ Here Δ_k is the orthogonal projection onto $\operatorname{Ker} \Theta^*(z_k)$.

$**$ Lemma 1.5 is contained essentially in (^{8}) for the case $\Theta = \psi I, \ \psi \in H^\infty$.

$***$) The system $\{\varphi_k\}_1^\infty$ is called uniformly minimal if the angles between φ_k and the linear span of all the remaining φ_n, $n \neq k$, are separated from zero, $k = 1, 2, \ldots$.

2.1. Lemma. The system $\{\psi_k\}_1^\infty$ is complete in K if and only if from $Jf = 0$, $f \in H^2(E)$, it follows that $f = \Theta g$, $g \in H^2(E)$.

2.2. Lemma. The following conditions are equivalent:

1. Any of the systems $\{\varphi_k\}_{i=1}^\infty$ (see (4)) is uniformly minimal.
2. $$\sup_k \left\| \pi_k \Theta_k^{-1}(z_k) \Delta_k \right\|_{\mathfrak{R}} < \infty. \tag{6}$$

2.3. Remark. If $\dim E = 1$, then condition (6) coincides with condition (C). In the general case, (C) always implies (6).

Denote by $l^2(\Delta_k)$ the set of all sequences $x = \{x\}_{k=1}^{\infty}$, $x_k \in \Delta_k E,\ k \leq 1$, such that $\sum_k \|x_k\|^2 < \infty$. We shall now establish that the possibility of interpolation in $H^2(E)$ of arbitrary sequences from $l^2(\Delta_k)$ is equivalent to the basis property of the system $\{\varphi_k\}_1^{\infty}$.

2.4. Theorem. If any system $\{\varphi_k\}_1^{\infty}$ is complete in K and uniformly minimal, then the following assertions are equivalent:

1. $JH^2(E) = l^2(\Delta_k)$.
2. $\{\varphi_k\}_1^{\infty}$ $(\{\psi_k\}_1^{\infty})$ is a Riesz basis in K.

3.a)

$$\sum_k (1 - |z_k|^2) \|\pi_k f(z_k)\|_E^2 < \infty, \qquad f \in H_0^2(E).$$

b)

$$\sum_k (1 - |z_k|^2) \|\Delta_k f(z_k)\|_E^2 < \infty, \qquad f \in H^2(E).$$

2.5. Corollary (7). If condition (C) is satisfied, then the system $\{\varphi_k\}_1^{\infty}$ $(\{\psi_k\}_1^{\infty})$ is a Riesz basis in K.

In conclusion we note that one can give (in terms of the function Θ) effective conditions for the expandability of vectors of K with respect to the system $\{\varphi_k\}_1^{\infty}$, even in the case when this system is not a Riesz basis.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
of the Academy of Sciences of the USSR

Leningrad State University
named after A. A. Zhdanov

Received
8 XII 1968

REFERENCES

[1] H. Helson, *Lectures on Invariant Subspaces*, N. Y., London, 1964.
[2] B. Sz.-Nagy, C. Foias, *Analyse harmonique des opérateurs de l' espace de Hilbert*, Budapest, 1967.
[3] M. S. Brodskii, M. S. Livshits, UMN, 13, issue 1 (1958).
[4] I. Ts. Gokhberg, M. G. Krein, *Introduction to the Theory of Linear Non-Self-Adjoint Operators in Hilbert Space*, "Nauka," 1965.
[5] I. M. Glazman, UMN, 13, issue 3, 179 (1958).
[6] A. S. Markus, DAN, 132, No. 3 (1960).
[7] V. E. Katsnelson, *Functional Analysis and Its Applications*, 1, issue 2 (1967).
[8] D. Sarason, Trans. Am. Math. Soc., 127, No. 2 (1967).
[9] H. S. Shapiro, A. L. Shields, Am. J. Math., 83, No. 3 (1961).

[10] L. Carleson, Am. J. Math., 80, No. 4 (1958).
[11] D. Newman, Trans. Am. Math. Soc., 92, 501 (1959).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.