Reports of the Academy of Sciences of the USSR

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1969. Volume 189, No. 4

UDC 519.212.3

MATHEMATICS

V. P. NOSKO

LOCAL STRUCTURE OF GAUSSIAN RANDOM FIELDS IN A NEIGHBORHOOD OF HIGH GLINTS

(Presented by Academician A. N. Kolmogorov on 25 IV 1969)

1. Let $\xi(t) = \xi(t, \omega)$, $t \in R^m$, $\omega \in (\Omega, \mathfrak{B}, P)$, be a real homogeneous (in the narrow sense) random field ([1]); R^m is m-dimensional Euclidean space; $(\Omega, \mathfrak{B}, P)$ is a probability space. Let $\{T_\tau, \tau \in R^m\}$ be a group of shift transformations in the space $(\Omega', \mathfrak{B}', P')$ of sample functions of the field $\xi(t, \omega)$, defined for each $\tilde{\omega} = z(t) \in \Omega'$ by the equality $\tilde{\omega}^\tau = T^\tau \tilde{\omega} = z(t + \tau)$.

Theorem 1. *If $\xi(t)$ is metrically transitive with respect to $\{T_\tau, \tau \in R^m\}$ (the definition is analogous to ([2])), is measurable ([3]), and its sample functions are, with probability 1, Riemann integrable on m-dimensional cubes $[0, T] \times \cdots \times [0, T]$, then for any functional η measurable on $(\Omega', \mathfrak{B}', P')$ with $\mathrm{M}|\eta| < \infty$*

$$\lim_{T \uparrow \infty} \frac{1}{T^m} \int_0^T \cdots \int_0^T \eta(\tilde{\omega}_\tau)\, d\tau_1 \dots d\tau_m = \mathrm{M}\eta$$

for almost all sample functions.

Corollary 1. *The assertion of Theorem 1 is valid for a real Gaussian homogeneous random field $\xi(t)$, $\mathrm{M}\xi(t) = 0$, whose sample functions are continuous with probability 1, and whose covariance function $\rho(\tau) = \mathrm{M}\xi(t)\xi(t + \tau) \to 0$ as $|\tau| \uparrow \infty$.*

The proof of the theorem consists in verifying the fulfillment of the conditions of Theorem VIII.7.17 from ([4]). Under the conditions of Corollary 1 the field $\xi(t)$ is metrically transitive; the proof is based on the methods of ([2]) (Chap. 7).

2. Let $\xi(t)$, $t \in R^m$, be a real random field satisfying the conditions of the paper ([5]), whose notation we shall adhere to below. In particular, for any fixed $\delta > 0$ and any bounded domain $\Gamma \subset R^m$,

$$P\left\{\max_{t',t''\in\Gamma,\ |t'-t''|\le h}\left|\ddot{\xi}_{ij}(t')-\ddot{\xi}_{ij}(t'')\right|>\delta\right\}=o(h^m)$$

as $h\downarrow 0$, for $i,j=1,\dots,m$. We shall also assume that glints of the field $\xi(t)$, aligned with the vector $(-b',1)$, are regular (^{5}), and that the flow of such glints is regular and has finite intensity.

Denote by $\eta^b_{t^1,\dots,t^n}(\Delta; A_0, C^0, \mathfrak{M}^i,\ i=1,\dots,n)$ the subflow of the flow of regular glints aligned with the vector $(-b',1)$ such that the glint at the point τ has height $\xi(\tau)\in A_0$, the values $\nabla\xi(\tau)\nabla'\in C^0$, and at the points $\tau+t^i$ the values $(\xi(\tau+t^i),\nabla\xi(\tau+t^i),\nabla\xi(\tau+t^i)\nabla')$, $i=1,\dots,n$, belong to the sets $\mathfrak{M}^i=A_i\times B^i\times C^i$, where $A_i\subset R^1$, $i=0,\dots,n$, B^i $(i=1,\dots,n)$ and C^i $(i=0,\dots,n)$ are parallelepipeds in R^m and in the space $\widetilde{G}_m$ of nonsingular symmetric matrices of size $m\times m$, respectively. Let $\lambda^b_{t,t^1,\dots,t^n}(A_0,C^0,\mathfrak{M}^i,\ (i=1,\dots,n))$ be the intensity of this subflow at the point t.

Lemma 1. Under the conditions stated at the beginning of Sec. 2,

$$\mu^b_{t,t^1,\dots,t^n}(A_0,C^0,\mathfrak{M}^i,\ i=1,\dots,n)=\int|\det\ddot{\mathbf{v}}|\,p^{(2)}_{t,t^1,\dots,t^n}(v,v_1,\dots,v_n,\mathbf{b},\dot{\mathbf{v}}^1,\dots,\dot{\mathbf{v}}^n,\ddot{\mathbf{v}},\ddot{\mathbf{v}}^1,\dots,\ddot{\mathbf{v}}^n)\,dv\,dv_1\cdots$$
$$\cdots dv_n\,d\dot{\mathbf{v}}^1\cdots d\dot{\mathbf{v}}^n$$
$$\{v\in A_0,\ \ddot{\mathbf{v}}\in C^0,\ (v_i,\dot{\mathbf{v}}^i,\ddot{\mathbf{v}}^i)\in\mathfrak{M}^i,\ i=1,\dots,n\}. \tag{1}$$

Let the flow of highlights coordinated with the vector $(-\mathbf{b}',1)$ be generated by the random set (^{6}), whose points $(\tau,u,\mathbf{b},\mathbf{v})$ correspond to highlights for which $\zeta_\tau=u$, $\nabla\zeta_\tau=\mathbf{b}'$, $\nabla\zeta_\tau\nabla'=\mathbf{v}=\|c_{kl}\|$, $k,l=1,\dots,m$, and for $\tau+t^i$, $i=1,\dots,n$,

$$(\zeta(\tau+t^i),\nabla\zeta(\tau+t^i),\nabla\zeta(\tau+t^i)\nabla')\in\mathfrak{M}^i.$$

In accordance with (1), the intensity of such a flow with respect to the measure that is the direct product of Lebesgue measures in the spaces R^m, R^1, and $R^{m(m+1)/2}$, is equal at the point t to

$$\mu^b_{t,t^1,\dots,t^n}(u,\ddot{\mathbf{v}},\mathfrak{M}^i,\ i=1,\dots,n)=\int|\det\ddot{\mathbf{v}}|\,p^{(2)}_{t,t^1,\dots,t^n}(u,v_1,\dots,v_n,\mathbf{b},\dot{\mathbf{v}}^1,\dots,\dot{\mathbf{v}}^n,\ddot{\mathbf{v}},\ddot{\mathbf{v}}^1,\dots,\ddot{\mathbf{v}}^n)$$
$$\times\,d\dot{\mathbf{v}}_1\cdots d\dot{\mathbf{v}}_n d\ddot{\mathbf{v}}^1\cdots d\ddot{\mathbf{v}}^n d\ddot{\mathbf{v}}^1\cdots d\ddot{\mathbf{v}}^n$$
$$\{(v_i,\dot{\mathbf{v}}^i,\ddot{\mathbf{v}}^i)\in\mathfrak{M}^i,\ i=1,\dots,n\}. \tag{2}$$

If in (2) we put $\mathfrak{M}^i=R^1\times R^m\times G_m$, then we obtain the intensity $\mu^b_t(u,\ddot{\mathbf{v}})$ at the point t of the flow of highlights coordinated with the vector $(-\mathbf{b}',1)$, for which $u,\mathbf{v}$ are, respectively, the values of the highlight height and of the matrix of second derivatives of the field at the highlight point. On the basis of

(1), (2), in the space of separable realizations having, at the points t, highlights coordinated with the vector $(-\mathbf{b}', 1)$, one may define coordinated systems of finite-dimensional probability distributions

$$\widetilde{P}^{b}_{t,t^1,\dots,t^n}(C^0, \mathfrak{M}^i,\ i = 1, \dots, n \mid A_0) = \frac{\mu^{b}_{t,t^1,\dots,t^n}(A_0, C^0, \mathfrak{M}^i,\ i = 1, \dots, n)}{\mu^{b}_{t}(A_0)}, \quad (3)$$

$$\widetilde{P}^{b}_{t,t^1,\dots,t^n}(\mathfrak{M}^i,\ i = 1, \dots, n \mid \zeta_t = u,\ \nabla\zeta_t\nabla' = \ddot{\mathbf{v}}) = \frac{\mu^{b}_{t,t^1,\dots,t^n}(u, \ddot{\mathbf{v}}, \mathfrak{M}^i,\ i = 1, \dots, n)}{\mu^{b}_{t}(u, \ddot{\mathbf{v}})}. \quad (4)$$

Moreover, by virtue of the homogeneity of the field $\zeta(t)$, the index t in the expressions obtained may be omitted. The coordinated system of conditional probabilities (4) ((3)) can be extended to a probability measure in the space of separable realizations, determining the conditional field $\tilde{\zeta}(t)$, for which $\zeta(0) = u,\ \nabla\tilde{\zeta}(0) = \mathbf{b},\ \nabla\tilde{\zeta}(0)\nabla' = \ddot{\mathbf{v}}$ $(\tilde{\zeta}(0) \in A_0,\ \nabla\zeta(0) = \mathbf{b})$. The probability distributions corresponding to (3), (4) will be called *ergodic* ([7]), and we shall mark them with a tilde, using this sign also to denote random variables and their moments if the latter correspond to ergodic probability distributions. Thus, the symbol for mathematical expectation $\widetilde{\mathbf{M}}^{b,A_0}$ corresponds to the probability distribution generated by (3), and $\widetilde{\mathbf{M}}^{b,u,\ddot{\mathbf{v}}}$ to that generated by (4).

The characteristics of random variables determined by the field $\tilde{\zeta}(t)$ admit a simple statistical interpretation.

Lemma 2. Let $\zeta(t)$ be a metrically transitive random field satisfying the conditions of Lemma 1. Then, for $\Delta_T = [0, T] \times \dots \times [0, T]$, with probability 1,

$$\widetilde{P}^{b,A_0}_{t^1,\dots,t^n}(C^0, \mathfrak{M}^i,\ i = 1, \dots, n) = \lim_{T\uparrow\infty} \frac{\eta^{b}_{t^1,\dots,t^n}(\Delta_T; A_0, C^0, \mathfrak{M}^i,\ i = 1, \dots, n)}{\eta^{b}(\Delta_T, A_0)}.$$

The assertion of the lemma is a consequence of metric transitivity and formula (3). For Gaussian fields, (2) and (4) imply the following.

Corollary 2. *The ergodic distributions (4) are Gaussian.*

3. Exceedances of a random field ζ, continuous with probability 1, above a level u are the portions of a realization of the field that pass not below the level u. The exceedance $F_u(\zeta_{t_0})$ above the level u, containing the fixed point $(t_0, \zeta_{t_0}) \in R^{m+1}$, $\zeta_{t_0} \geq u$, of the realization is called the set of all such points (t, ζ_t), $\zeta_t \geq u$, of this realization that can be joined to (t_0, ζ_{t_0}) by a continuous curve all of whose points are situated not below the level u.

Theorem 2. *Let $\zeta(t) = \zeta(t_1, t_2, \omega)$ be a metrically transitive Gaussian random field, $M\zeta(t) \equiv 0$, whose covariance function $\rho(t)$ has all fourth derivatives at zero and for $|t| \le h_0$*

$$\left|\frac{\partial^4 \rho(t)}{\partial t_1^{\varepsilon_1} \partial t_2^{\varepsilon_2}} - \frac{\partial^4 \rho(0)}{\partial t_1^{\varepsilon_1} \partial t_2^{\varepsilon_2}}\right| < \psi(|t_1|, |t_2|); \qquad \varepsilon_1, \varepsilon_2 = 0, \dots, 4, \quad \varepsilon_1 + \varepsilon_2 = 4,$$

where $\psi(t_1, t_2)$, $\psi(0,0) = 0$, is a continuous function, nondecreasing in $|t_1|$ and in $|t_2|$, ensuring fulfillment of the conditions of the preceding item. Then, with probability arbitrarily close to 1 as $u \uparrow \infty$, the exceedance $F_u(\zeta(x_0, y_0))$, where $\zeta(x_0, y_0) = u$, $\nabla\zeta'(x_0, y_0) = (b_1, b_2)$, is, up to infinitesimals of order $o(1/u)$, a segment of an elliptic paraboloid lying above the plane $z = u$, whose vertex is at the point

$$\left(x_0 + \bar{t}_1,\, y_0 + \bar{t}_2,\, u - \frac{\rho(0)}{2u} \frac{[\rho''_{11}(0) b_2^2 - 2\rho''_{12}(0) b_1 b_2 + \rho''_{22}(0) b_1^2]}{[\rho''_{11}(0)\rho''_{22}(0) - (\rho''_{12}(0))^2]}\right),$$

the axis is parallel to the axis oz, the principal sections are rotated with respect to the coordinate planes $t_1 oz$ and $t_2 oz$ by an angle α, and the parameters are p and q. Here

$$\bar{t}_1 = \frac{\rho(0)}{u} \frac{[\rho''_{12}(0) b_2 - \rho''_{22}(0) b_1]}{[\rho''_{11}(0)\rho''_{22}(0) - (\rho''_{12}(0))^2]}, \qquad \bar{t}_2 = \frac{\rho(0)}{u} \frac{[\rho''_{12}(0) b_1 - \rho''_{11}(0) b_2]}{[\rho''_{11}(0)\rho''_{22}(0) - (\rho''_{12}(0))^2]},$$

the angle α is determined from the conditions

$$\operatorname{tg} 2\alpha = \frac{2\rho''_{12}(0)}{\rho''_{11}(0) - \rho''_{22}(0)}, \qquad \operatorname{sign}(\sin 2\alpha) = \operatorname{sign}(\rho''_{12}(0)),$$

$$p = -\frac{2\rho(0)}{u}\left[\rho''_{11}(0) + \rho''_{22}(0) + \sqrt{(\rho''_{11}(0) - \rho''_{22}(0))^2 + 4(\rho''_{12}(0))^2}\right],$$

$$q = -\frac{2\rho(0)}{u}\left[\rho''_{11}(0) + \rho''_{22}(0) - \sqrt{(\rho''_{11}(0) - \rho''_{22}(0))^2 + 4(\rho''_{12}(0))^2}\right],$$

$$\rho''_{ij} = \partial^2 \rho(0)/\partial t_i \partial t_j, \qquad i, j = 1, 2.$$

The proof is connected with estimating

$$\widetilde{M}^{b,u,\ddot{v}(u)} \tilde{\zeta}_i(t_1, t_2) \quad \text{and} \quad \tilde{\zeta}_i^{b,u,\ddot{v}(u)}(t_1, t_2) - \widetilde{M}^{b,u,\ddot{v}(u)} \tilde{\zeta}_i(t_1, t_2).$$

An asymptotic representation is also used for the values of the second derivatives of the field at points t_0 for which $\zeta(t_0) = u$. As the function $\psi(t_1, t_2)$ one may take the function $\psi(t_1, t_2) = c(|t_1|^\delta + |t_2|^\delta)$, $c > 0$, $\delta > 0$ (this follows from (5)).

Consider the following functionals of realizations of the field $\zeta(t_1, t_2)$: V_u is the volume of the body bounded above by the surface of the exceedance above the level u, and below by the plane $z = u$; S_u is the area of the base of this body; m_u is its height.

Corollary 3. *As $u \uparrow \infty$, the distributions of the random variables um_u, $\chi u^2 S_u$, $(2\chi u^3 V_u)^{1/2}$, where $\chi = [\rho''_{11}(0)\rho''_{22}(0) - (\rho''_{12}(0))^2]^{1/2} \cdot [2\pi\rho(0)]^{-1}$, are equivalent to the distribution of the random variable $u\widetilde{m}_u$, where $\widetilde{m}_u$ is the height of a segment of an elliptic paraboloid approximating the exceedance. In co-*

in accordance with item 2, the ergodic distribution of $\tilde{u}\tilde{m}_u$ is

$$\tilde{P}^0(\tilde{u}\tilde{m}_u < v \mid \tilde{m}_u \geq 0) = \int_u^{u+(v/u)} du \int_{\ddot{v} \in G_2} \mu_t^0(u, \ddot{v})\, d\ddot{v} \Big/ \int_u^\infty du \int_{\ddot{v} \in G_3} \mu_t^0(u, \ddot{v})\, d\ddot{v}.$$

The present work is closely related to [7]. The author considers it a pleasant duty to express his gratitude to Yu. K. Belyaev for posing the problem.

Moscow State University
named after M. V. Lomonosov

Received
18 IV 1969

REFERENCES CITED

1. A. M. Yaglom, *Uspekhi Matematicheskikh Nauk*, **7**, no. 5, 3 (1952).
2. H. Cramér, M. R. Leadbetter, *Stationary and Related Stochastic Processes*, N.Y., 1967.
3. I. I. Gikhman, A. V. Skorokhod, *Introduction to the Theory of Random Processes*, "Nauka," 1965.
4. N. Dunford, J. Schwartz, *Linear Operators, General Theory*, IL, 1962.
5. Yu. K. Belyaev, *Doklady Akademii Nauk*, **176**, no. 3 (1967).
6. Yu. K. Belyaev, *Theory of Probability and Its Applications*, **13**, 3, 578 (1968).
7. Yu. K. Belyaev, V. P. Nosko, ibid., **14**, 2 (1969).