EXPLICIT REPRESENTATION AND ASYMPTOTICS AS \(t \to \infty\) OF THE SOLUTION OF THE CAUCHY PROBLEM FOR THE LINEARIZED SYSTEM OF A ROTATING COMPRESSIBLE FLUID

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Abstract

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EXPLICIT REPRESENTATION AND ASYMPTOTICS AS $t \to \infty$ OF THE SOLUTION OF THE CAUCHY PROBLEM FOR THE LINEARIZED SYSTEM OF A ROTATING COMPRESSIBLE FLUID

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Consider the system

$$\partial \mathbf{v}/\partial t - [\mathbf{v}, \mathbf{k}] + \operatorname{grad} p = \mathbf{F}(x,t), \qquad \alpha^2 \partial p/\partial t + \operatorname{div} \mathbf{v} = \psi(x,t) \tag{1}$$

in the domain $\{x \in E_3,\ t \geq 0\}$, where $\mathbf{v}(x,t) = (v_1, v_2, v_3)$; $\mathbf{k} = (0,0,1)$; $\mathbf{F}(x,t) = (F_1, F_2, F_3)$; $[\cdot,\cdot]$ denotes the vector product; $\alpha^2 = \text{const}$ will conventionally be called the coefficient of compressibility. System (1) is hyperbolic and has multiple characteristics.

In our papers (^{3},^{4}) the Cauchy problem and boundary-value problems in bounded domains for system (1) were studied. For $\alpha = 0$ system (1) was first considered by S. L. Sobolev (^{1},^{2}). Applications of systems of the form (1) are considered in (^{5}).

In the present paper we study the asymptotic behavior as $t \to \infty$ of the solution of the Cauchy problem with initial data

$$\mathbf{v}|_{t=0} = \mathbf{v}^0(x); \qquad p|_{t=0} = p^0(x) \tag{2}$$

for the homogeneous system (1), and find the rate at which the solution decreases as $t \to \infty$, provided that the initial data (2) are smooth and decrease well as $|x| \to \infty$ (for example, are finite), and at the same time we give an explicit representation of this solution.

In paper (^{3}) we constructed explicitly the solution of problem (1), (2) in the form of singular integrals with kernels having strong singularities. In this case the initial data and the right-hand sides of (1) could be nondifferentiable functions.

At the same time, the solution of the Cauchy problem was constructed for an equation of fourth order of the form

$$\partial^2 \Delta u/\partial t^2 + \partial^2 u/\partial x_3^2 - \alpha^2(\partial^4 u/\partial t^4 + \partial^2 u/\partial t^2) = f(x,t) \tag{3}$$

(to which each function of system (1) satisfies) with initial conditions

$$\partial^k u/\partial t^k\big|_{t=0} = u_k(x); \qquad k = 0, 1, 2, 3. \tag{4}$$

The solution of problem (3), (4) contained integrals with weak singularities of the kernels, and a certain smoothness was required of the initial data (4).

In the present paper we assume that the initial data (2) and the right-hand sides of system (1) are sufficiently smooth (let $\mathbf{v}^0$ have second derivatives, p^0 first derivatives, and the functions $\mathbf{F}$ and ψ be three times differentiable). In this case the representation of the solution of the Cauchy problem for system (1) has the form

$$\mathbf{v}(x,t) = \frac{1}{4\pi}\iint_{r=t/\alpha}\left\{-\frac{1}{r}\frac{\partial \mathbf{v}_0}{\partial n} + \left(\frac{1}{r^2} - \frac{\alpha^2\rho^2}{2r^2}\right)\mathbf{v}^0 + \frac{\alpha}{r}\left([\mathbf{v}^0, \mathbf{k}] - \operatorname{grad} p^0\right)\right\} ds_y +$$

$$+\frac{1}{4\pi}\iiint_{r\le t/\alpha}\left\{G(x-y,t)\frac{\partial \vec{\Phi}_0(y)}{\partial y_3} + \sum_{k=1}^{3}\frac{\partial^{kG}(x-y,t)}{\partial t^k}\vec{\Phi}_k(y)\right\} dy +$$

$$+\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\int\limits_0^{t-\alpha r} G(x-y,t-\tau)\mathbf{f}(y,\tau)\,d\tau\,dy; \tag{5}$$

$$p(x,t) = \frac{1}{4\pi}\iint\limits_{r\le t/\alpha}\left\{-\frac{1}{r}\frac{\partial p^0(y)}{\partial n} - \frac{1}{\alpha r}\operatorname{div}\mathbf{v}^0(y) + \left(\frac{1}{r^2} - \frac{\alpha^2\rho^2}{2r^2}\right)p^0(y)\right\} ds_y +$$

$$+\frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left\{G(x-y,t)\frac{\partial v_3^0}{\partial y_3} + \frac{\partial G}{\partial t}\left(\frac{\partial v_1^0}{\partial y_2} - \frac{\partial v_2^0}{\partial y_1} + \alpha^2 p^0\right) + \frac{\partial^2 G}{\partial t^2}\operatorname{div}\mathbf{v}^0 +\right.$$

$$\left.+\frac{\partial^3 G}{\partial t^3}\alpha^2 p^0\right\} dy + \frac{1}{4\pi}\iiint\limits_{r\le t/\alpha}\left[\int\limits_0^{t-\alpha r} G(x-y,t-\tau) f_4(y,\tau)\,d\tau\right] dy, \tag{6}$$

where the vector-functions $\vec{\Phi}_k(y)$ in (5) are expressed in terms of the initial data:

$$\vec{\Phi}_0(y) = -\operatorname{rot}\mathbf{v}^0 + \alpha^2\mathbf{k}p^0;$$

$$\vec{\Phi}_1(y) = -\Delta\mathbf{v}^0 + \operatorname{grad}\operatorname{div}\mathbf{v}^0 - \alpha^2[\operatorname{grad}p^0, \mathbf{k}] + \alpha^2\mathbf{k}v_3^0;$$

$$\vec{\Phi}_2(y) = \alpha^2(\operatorname{grad}p^0 - [\mathbf{v}^0, \mathbf{k}]); \qquad \vec{\Phi}_3(y) = \alpha^2\mathbf{v}^0;$$

$$G(x-y,t) = \frac{1}{\rho}\int\limits_0^{\rho\sqrt{t^2-\alpha^2r^2}/r} \frac{\eta}{\sqrt{\eta^2+\alpha^2\rho^2}} J_0(\eta)\,d\eta,$$

where

$$\rho^2 = \sum_{i=1}^{2}(x_i - y_i)^2, \qquad r^2 = \rho^2 + (x_3 - y_3)^2;$$

J_0 is the Bessel function of order zero. The functions $\mathbf{f}$ and f_4 are expressed in terms of $\mathbf{F}$ and ψ:

$$\mathbf{f}(y,\tau) = \left(-\operatorname{rot}\frac{\partial\mathbf{F}}{\partial y_3} + \mathbf{k}\frac{\partial\psi}{\partial y_3}\right) + \left(\Delta\frac{\partial\mathbf{F}}{\partial\tau} - \operatorname{grad}\operatorname{div}\frac{\partial\mathbf{F}}{\partial\tau} + \right.$$
$$\left. + \left[\operatorname{grad}\frac{\partial\psi}{\partial\tau}, \mathbf{k}\right] - \alpha^2\mathbf{k}\frac{\partial F_3}{\partial\tau}\right) + \left(\operatorname{grad}\frac{\partial^2\psi}{\partial\tau^2} - \alpha^2\left[\frac{\partial^2\mathbf{F}}{\partial\tau^2}, \mathbf{k}\right]\right) - \alpha^2\frac{\partial^3\mathbf{F}}{\partial\tau^3},$$

$$f_4(y,\tau) = \frac{\partial F_3}{\partial y_3} + \frac{\partial^2 F_2}{\partial y_1\partial\tau} - \frac{\partial^2 F_1}{\partial y_2\partial\tau} - \frac{\partial\psi}{\partial\tau} + \operatorname{div}\frac{\partial^2\mathbf{F}}{\partial\tau^2} - \frac{\partial^3\psi}{\partial\tau^3}.$$

Let us note that the solution (5), (6) for $\alpha \to 0$ passes into the solution of the non-hyperbolic system (1), in which the last equation has the form $\operatorname{div}\mathbf{v} = \psi$. Such a solution with weak singularities of the kernels for an incompressible ($\alpha = 0$) fluid was constructed by us in [6] by the Fourier method.

Theorem 1. *The solution of problem (1), (2) is unique in* $L_2(Q)$, *where* $Q = \{x \in E_3,\ 0 \le t \le T\}$.

The proof follows from the energy estimate [4]

$$\int_Q (\mathbf{v}^2 + p^2)\,dx\,dt \le C\left\{\int_Q (\mathbf{F}^2 + \psi^2)\,dx\,dt + \int_{E_3} (\mathbf{v}^{02} + p^{02})\,dx\right\},$$

which holds for problem (1), (2). Let us note that under our assumptions the solution under consideration certainly belongs to L_2.

The representation (5), (6) makes it possible to obtain a number of interesting properties of the solution of the Cauchy problem; in particular, it implies the finiteness of the velocity of propagation of disturbances, which in the present case is equal to $1/\alpha$; it also makes it possible to study the behavior of the solution of the Cauchy problem as $t \to \infty$.

Let us consider the solution (5) for the homogeneous system (1) ($\mathbf{f}, f_4 \equiv 0$); making the substitution $x - y = \xi,\ dy = d\xi$, we integrate by parts with respect to ξ_3 the first term in the second integral of formula (5), taking into account that on the surface of the cone $G = 0$. Using the Bessel equation $J_0(\eta) = -J_0''(\eta) - J_0'(\eta)/\eta$, the formula for the vector $\mathbf{v}(x,t)$—the solution of the Cauchy problem for the homogeneous system—can be written in the form:

$$\mathbf{v}(x,t) = \frac{1}{4\pi} \iint\limits_{r=t/\alpha} \left\{ -\frac{1}{r}\frac{\partial \mathbf{v}^0}{\partial n} + \left(\frac{1}{r^2} - \frac{\alpha^2\rho^2}{2r^2} \right) \mathbf{v}^0 + \frac{\alpha}{r} \left([\mathbf{v}^0, \mathbf{k}] - \operatorname{grad} p^0 \right) \right\} ds +$$

$$+ \frac{1}{4\pi} \iiint_{r \le t/a} \left\{ \left(-\frac{t\xi_3}{r^3} J_0'' - \frac{t\xi_3}{r^2 \rho \sqrt{t^2 - a^2r^2}} J_0' \right) \vec{\Phi}_0(x+\xi) + \right.$$

$$+ \left(-\frac{1}{r} J_0'' - \frac{1}{\rho\sqrt{t^2 - a^2r^2}} J_0' \right) \vec{\Phi}_1(x+\xi) + \frac{\rho t}{r^2\sqrt{t^2 - a^2r^2}} J_0 \vec{\Phi}_2(x+\xi) +$$

$$\left. + \left(\frac{\rho^2 t^2}{r^3(t^2 - a^2r^2)} J_0'' - \frac{a^2\rho}{(t^2 - a^2r^2)^{3/2}} J_0' \right) \vec{\Phi}_3(x+\xi) \right\} d\xi, \qquad (7)$$

where the omitted arguments of the Bessel functions are equal to $\rho\sqrt{t^2 - a^2r^2}/r$, and now $r^2 = \rho^2 + \xi_3^2$, $\rho^2 = \xi_1^2 + \xi_2^2$. An analogous form (with the same potentials, but with other $\vec{\Phi}_i$) will also be possessed by the formula for p.

We note that all potentials $G_i(\xi, t)$ (as we shall now denote the coefficients of $\vec{\Phi}_i$ in (7)) will have weak singularities on the surface of the cone.

Theorem 2. *If the initial data (2) belong to C_0^∞, then the solution $\mathbf{v}(x,t), p(x,t)$ of the Cauchy problem for the homogeneous system (1) decreases as $t \to \infty$ like $C(x)/t$, where $C(x)$ is a bounded function.*

Proof. If t is sufficiently large and the initial data are finite, then the first integral in (7) will be equal to zero. Transform the first term in the second integral of formula (7). We have

$$-\frac{1}{4\pi} \iiint_{r \le t/a} \frac{t\xi_3}{r^3} J_0'' \left(\frac{\rho\sqrt{t^2 - a^2r^2}}{r} \right) \vec{\Phi}_0(x+\xi)\, d\xi =$$

$$= -\frac{1}{4\pi}\iiint_{r\le t/a} J_0'\left(\frac{\rho\sqrt{t^2-a^2r^2}}{r}\right)\frac{\partial}{\partial\xi_3}\left(\frac{\sqrt{t^2-a^2r^2}}{\rho t}\vec{\Phi}_0(x+\xi)\right)d\xi.$$

Then

$$\frac{1}{4\pi}\iiint_{r\le t/a} G_0(\xi,t)\vec{\Phi}_0(x+\xi)\,d\xi =$$

$$= -\frac{1}{4\pi}\iiint_{r\le t/a}\left[\frac{\partial\vec{\Phi}_0}{\partial\xi_3}\frac{\sqrt{t^2-a^2r^2}}{\rho t}J_0'\left(\frac{\rho\sqrt{t^2-a^2r^2}}{r}\right)-\right.$$

$$\left.-\frac{\xi_3\sqrt{t^2-a^2r^2}}{\rho t r^2}\vec{\Phi}_0 J_0'\left(\frac{\rho\sqrt{t^2-a^2r^2}}{r}\right)\right]d\xi = Q_1+Q_2.$$

Transform Q_2. Passing to spherical coordinates, we have

$$Q_2 = -\frac{1}{4\pi t}\int_0^{2\pi}d\varphi\int_0^{t/a}dr\int_0^{\pi}\vec{\Phi}_0\,dJ_0 = \frac{1}{2t}\int_0^{t/a}\left[\vec{\Phi}_0(x_1,x_2,x_3+r)-\right.$$

$$\left.-\vec{\Phi}_0(x_1,x_2,x_3-r)\right]dr + \frac{1}{4\pi t}\int_0^{2\pi}d\varphi\int_0^{t/a}dr\int_0^{\pi}J_0\left(\sin\theta\sqrt{t^2-a^2r^2}\right)\frac{d\vec{\Phi}_0}{d\theta}\,d\theta,$$

whence $Q_2 = C(x)/t$, where $C(x)$ is uniformly bounded as $|x|\to\infty$.

Transform Q_1. Denote the smooth finite function $\partial\vec{\Phi}_0/\partial\xi_3$ by $\vec{\Phi}_{03}$, and expand it in a Taylor series in a neighborhood of $\xi_3=0$, with the remainder term in integral form (the first two arguments $x_1+\xi_1, x_2+\xi_2$ of the function $\vec{\Phi}_{03}$ will be omitted for brevity):

$$\vec{\Phi}_{03}(x_3+\xi_3)-\vec{\Phi}_{03}(x_3) = \left(\frac{\partial\vec{\Phi}_{03}}{\partial\xi_3}\right)_{\xi_3=0}\xi_3 + \int_{x_3}^{x_3+\xi_3}(x_3+\xi_3-\eta)\frac{\partial^2\vec{\Phi}_{03}(\eta)}{\partial\eta^2}\,d\eta.$$

Then

$$Q_1 = -\frac{1}{4\pi t}\iiint_{r\le t/a}\frac{\sqrt{t^2-a^2r^2}}{\rho}J_0'\left(\frac{\rho\sqrt{t^2-a^2r^2}}{r}\right)\left[\vec{\Phi}_{03}(x_3)+\left(\frac{\partial\vec{\Phi}_{03}}{\partial\xi_3}\right)_{\xi_3=0}\xi_3+\right.$$

$$+\int_{x_3}^{x_3+\xi_3}(x_3+\xi_3-\eta)\frac{\partial^2\vec{\Phi}_{03}(\eta)}{\partial\eta^2}\,d\eta\Bigg]\,d\xi=\sum_{i=1}^{3}Q_{1i}.$$

Consider Q_{11}. In spherical coordinates, after the substitution $\sin\theta=\gamma$, $d\theta=d\gamma/\sqrt{1-\gamma^2}$, we have

$$Q_{11}=-\frac{1}{2\pi t}\int_0^{2\pi}d\varphi\int_0^{t/a}r\,dr\int_0^1\frac{\sqrt{t^2-a^2r^2}\,J_0'\big(\gamma\sqrt{t^2-a^2r^2}\big)}{\sqrt{1-\gamma^2}}\times$$

$$\times\vec{\Phi}_{03}(x_1+r\gamma\cos\varphi,\ x_2+r\gamma\sin\varphi,\ x_3)\,d\gamma.$$

Since $\vec{\Phi}_{03}$ depends differentiably on γ, expanding $\vec{\Phi}_{03}$ in a Taylor series in a neighborhood of $\gamma=1$, we obtain

$$\vec{\Phi}_{03}(\gamma)-\vec{\Phi}_{03}(1)=\left(\frac{\partial\vec{\Phi}_{03}}{\partial\gamma}\right)_{\gamma=1}(\gamma-1)+o((\gamma-1)^2).$$

This makes it possible to integrate by parts the improper integral in γ and to get rid of the factor $\sqrt{t^2-a^2r^2}$ in the numerator, while the remaining integral with $\vec{\Phi}_{03}(1)$ can be computed explicitly. Indeed, we have

$$Q_{111}=-\frac{1}{2\pi t}\int_0^{2\pi}d\varphi\int_0^{t/a}r\,dr\int_0^1\frac{\sqrt{t^2-a^2r^2}\,J_0'\big(\gamma\sqrt{t^2-a^2r^2}\big)}{\sqrt{1-\gamma^2}}\times$$

$$\times\vec{\Phi}_{03}(x_1+r\cos\varphi,\ x_2+r\sin\varphi,\ x_3)\,d\gamma.$$

But (see (2))

$$\frac{1}{t}\int_0^1\sqrt{t^2-a^2r^2}\,\frac{J_0'\big(\gamma\sqrt{t^2-a^2r^2}\big)}{\sqrt{1-\gamma^2}}\,d\gamma=\frac{\cos\sqrt{t^2-a^2r^2}-1}{t}.$$

Therefore we have

$$Q_{111}=\frac{1}{4\pi t}\int_0^{2\pi}d\varphi\int_0^{t/a}r\left[1-\cos\sqrt{t^2-a^2r^2}\right]\times$$

$$\times\vec{\Phi}_{03}(x_1+r\cos\varphi,\ x_2+r\sin\varphi,\ x_3)\,dr.$$

Thus we obtain that $Q_{11}=C(x)/t$. Notice that for $a=0$ (see (6)) we had $Q_{11}=\frac{1-\cos t}{t}C(x)$, since there was no retardation of the argument there. In the integrals Q_{12} and Q_{13}, passing to spherical coordinates, one can carry out integration by parts with respect to θ and obtain that they are also representable in the form $C(x)/t$, where in all cases $C(x)$ is uniformly bounded in $x \in E_3$.

In an analogous way, transforming the terms with the potentials $G_1 - G_3$ in formula (7), we obtain the proof of Theorem 2.

Remark 1. In the proof of Theorem 2 it is used only that the initial data belong to C^5 and decrease sufficiently rapidly as $|x| \to \infty$.

Remark 2. We have obtained the same rate of decrease as $t \to \infty$ in the cases of a compressible and an incompressible fluid. Thus, the principal terms in the asymptotics are determined not by one of the higher terms $a^2 \partial\rho/\partial t$ in the system, but by the lower-order term $[\vec{v}, \vec{k}]$. This phenomenon is connected with the presence of multiple characteristics in the system.

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Note: Figure translations are in progress. See original paper for figures.

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