THE LOCALIZATION PRINCIPLE FOR ALMOST THE ENTIRE SEQUENCE OF PARTIAL SUMS OF A FOURIER SERIES WITH RESPECT TO A FUNDAMENTAL SYSTEM OF FUNCTIONS

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Abstract

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THE LOCALIZATION PRINCIPLE FOR ALMOST THE ENTIRE SEQUENCE OF PARTIAL SUMS OF A FOURIER SERIES WITH RESPECT TO A FUNDAMENTAL SYSTEM OF FUNCTIONS

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In paper ([1]) we established that, in the classical formulation of the question of the localization principle for Fourier series with respect to the so-called fundamental system of functions (f.s.f.) of the Laplace operator in an arbitrary N-dimensional domain g *, one inevitably has to require that the function $f(x)$ being expanded, even at points "far" from the point x_0 under consideration, satisfy high smoothness requirements, increasing without bound as the number N of dimensions increases. Thus, if $N \geqslant 2$, the fact that the function $f(x)$ is finite in an arbitrary N-dimensional domain g and vanishes in a neighborhood $\mathcal{D}$ of the point x_0 under consideration, while belonging throughout the domain g to the class $C^{([N/2]-1,\alpha)}$ for any α smaller than the number $3/4-(-1)^N \cdot 1/4$, is still insufficient for convergence of the Fourier series of the indicated function at the point x_0.

It is natural to arrive at the idea of departing from the classical formulation of the question of the localization principle, and this is what we do in the present paper.

In our work, as usual, we shall assume that the function $f(x)$ being expanded vanishes in some neighborhood $\mathcal{D}$ of the point x_0 under consideration. But we shall study conditions ensuring not the convergence of the Fourier series of this function at the point x_0 itself, but rather the convergence almost everywhere of this series in the neighborhood $\mathcal{D}$ of the indicated point. Moreover, we shall study the question of convergence almost everywhere in $\mathcal{D}$ not necessarily of the whole sequence, but only of "almost the entire" sequence of partial sums of the Fourier series.

We succeed in proving that if the function $f(x)$ in the entire N-dimensional domain g belongs only to the class L_2, then for almost all points x of that domain $\mathcal{D}$ where $f(x)$ vanishes, almost the entire sequence $\{S_n(x)\}$ of partial sums of the Fourier series of this function not only tends to zero, but also has the order

$$S_n(x) = o\left(\frac{\log^{1/2}(1+n)}{n^{1/2N}}\right), \tag{1}$$

and for a certain class of domains and f.s.f. (including multiple trigonometric Fourier series) even the order

$$S_n(x) = o(1/n^{1/2N}). \tag{2}$$

This result is new even for multiple trigonometric Fourier series (with spherical partial sums). Moreover, even for a one-dimensional trigonometric Fourier series, when partial sums–

* By a complete orthonormalized system of functions $\{u_n(x)\}$ in an N-dimensional domain g we mean a **fundamental system of functions** of the Laplace operator in this domain if every function $u_n(x)$ belongs in the open domain g to the class $C^{(2)}$ and, for some nonnegative number λ_n, satisfies inside g the equation $\Delta u_n + \lambda_n u_n = 0$. In this case the numbers $\{\lambda_n\}$ will be called **fundamental numbers** (see ([1]), p. 1, § 3, Ch. I).

are explicitly written out by means of the Dirichlet integral, and the estimate (2) can be established by elementary means; we have not found publications to which one could refer concerning this estimate.

1°. We pass to the precise formulation of the results.

Definition 1. We shall say that a subsequence $\{S_{n_k}\}$ $(k = 1, 2, ...)$ of a numerical sequence $\{S_n\}$ has **density one** if there exists the limit, equal to one, as $m \to \infty$, of the fraction $n(m)/m$, where $n(m)$ denotes the number of indices n_k not exceeding m.

Definition 2. We shall say that **almost the whole** sequence $\{S_n\}$ has property A if from this sequence one can select a subsequence of density one having property A.

In particular, if by property A one understands convergence, we arrive at the notion of convergence of almost the whole sequence.

Many mathematicians have studied the convergence of almost the whole sequence of partial sums of orthogonal Fourier series (see ([2–7]) and pp. 270–282 of the monograph ([8])).

In the present paper we shall consider an arbitrary N-dimensional domain g and, in it, an arbitrary orthonormal system of eigenfunctions of the Laplace operator, with respect to which we require only that for the eigenvalues $\{\lambda_n\}$ either the asymptotic estimate

$$\sum_{\lambda_k \le \lambda} 1 = C\lambda^{N/2} + O(\lambda^{(N-1)/2}), \qquad \text{where } C = \text{const}, \tag{A}$$

or at least the weaker asymptotic estimate*

$$\sum_{\lambda_k \le \lambda} 1 = C\lambda^{N/2} + O(\lambda^{(N-1)/2} \log(2+\lambda)) \tag{B}$$

hold.

Main theorem. *If g is an arbitrary N-dimensional domain; $\mathfrak{D}$ is any of its subdomains; the function $f(x)$ belongs to the class L_2 in the whole domain g and vanishes in the domain $\mathfrak{D}$; and if, for the orthonormal system of eigenfunctions of the Laplace operator, estimate (A) (estimate (B)) is valid, then for almost all points x of the domain $\mathfrak{D}$, almost the whole sequence $\{S_n(x)\}$ of partial sums of the Fourier series of the function $f(x)$ not only tends to zero, but has order (2) (respectively (1)).*

Remark 1. The basic estimate (2), in its order with respect to n, is final in the sense that the quantity $o(1/n^{1/2N})$ standing on the right-hand side of (2) cannot be replaced by $O(1/n^{1/2N+\varepsilon})$, whatever the previously prescribed positive number ε may be.

Remark 2. We must stipulate that for almost every point x of the domain $\mathfrak{D}$, from the sequence $\{S_n(x)\}$ of partial sums one should choose its own (generally speaking, depending on x) subsequence of density one, for whose elements estimate (2) (or, respectively, (1)) is realized.

2°. We now indicate the scheme of the proof of the main theorem. The following two lemmas are the principal tool in the proof.

Lemma 1. *Let all the conditions of the main theorem be fulfilled; $\mathfrak{D}'$ denotes any strictly interior subdomain of the domain $\mathfrak{D}$; the positive number R is less than the minimum distance between the boundaries of $\mathfrak{D}$ and $\mathfrak{D}'$; p is any number equal to $0, 1, 2, \ldots, [N/2]$.*

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* We note that for concrete systems of eigenfunctions of the Laplace operator with boundary conditions of the first, second, or third kinds, estimate (B) is in any case valid (see ([9]), vol. I, pp. 375-377), while for the particular case of the N-dimensional cube and the corresponding multiple trigonometric Fourier series estimate (A) is valid.

Then for almost all points x of the domain $\mathcal{D}'$ the series*

$$\sum_{k=1}^{\infty} f_k u_k(x) \frac{J_{N/2-p}(R\sqrt{\lambda_k})}{\sqrt{\lambda_k}^{N/2-p}} \tag{3}$$

converges to zero.

The very fact of almost-everywhere convergence of the series (3) follows trivially from D. E. Menshov' s theorem. The main difficulty is to prove that this series converges almost everywhere **precisely to zero**.

It suffices to prove that, whatever measurable set E contained in $\mathcal{D}'$ may be, the series (3), after being integrated over E, converges to zero. For $p = 0$ the latter assertion is obtained without particular difficulty from the mean-value formula for $u_k(y)$ and from the convergence for $p = 0$ of the series (3) in L_2. For $p = 1, 2, \ldots, [N/2]$ one first establishes that the series (3), integrated term by term over E, converges uniformly in R on the segment $0 < \varepsilon \leq R \leq R_0$, where R_0 is any number smaller than the minimum distance between the boundaries $\mathcal{D}$ and $\mathcal{D}'$, and then observes that the indicated series is obtained from the corresponding series for $p = 0$ by termwise application of the operation D^p, where

$$D\psi = \frac{d}{dR}\left(\frac{\psi}{R}\right).$$

With the help of Lemma 1 the following fundamental lemma is proved.

Lemma 2. *If all the conditions of the main theorem are fulfilled and $S_n(x)$ denotes the n-th partial sum of the Fourier series of the function $f(x)$, then almost everywhere in the domain $\mathcal{D}$ the series*

$$\sum_{\lambda_n \geq 1} \frac{S_n^2(x)}{\lambda_n^{(N-1)/2}} \left[\sum_{\lambda_n \geq 1} \frac{S_n^2(x)}{\lambda_n^{(N-1)/2} \log(1+\lambda_n)}\right] \tag{4}$$

converges.

The proof of Lemma 2 requires some technique, the separate stages of which were developed in our papers ([1,10]). With the help of this technique and Lemma 1, for almost all points x of an arbitrary subdomain $\mathcal{D}'$ of the domain $\mathcal{D}$, one succeeds in representing $S_n(x)$ in the form

$$S_n(x) = \frac{1}{2} \sum_{\substack{k \leq n \\ \lambda_k = \lambda_n}} f_k u_k(x) - \frac{1}{2} \sum_{\substack{k > n \\ \lambda_k = \lambda_n}} f_k u_k(x) + \sum_{k=1}^{\infty} f_k u_k(x) \frac{\sqrt{\lambda_n}^{\,N/2-[N/2]}}{\sqrt{\lambda_k}^{\,N/2-[N/2]-1}} I_k^n(R), \tag{5}$$

where

$$I_k^n(R) = \int_R^\infty J_{N/2-[N/2]}\left(r\sqrt{\lambda_n}\right) J_{N/2-[N/2]-1}\left(r\sqrt{\lambda_k}\right) dr, \tag{6}$$

and R is any number smaller than the minimum distance between the boundaries of the domains $\mathcal{D}$ and $\mathcal{D}'$.

To complete the proof of Lemma 2, one must prove the convergence, after integration over the domain $\mathcal{D}'$, of the series (4), and use here equality (5) and the widely known estimate for the integral (6).

With the help of Lemma 2 the main theorem is proved elementarily, for from this lemma it follows that for almost all points x of the domain $\mathcal{D}$ the sequence $\{S_n(x)n^{1/2N}\}$ (respectively,

$$\left\{S_n(x)\frac{n^{1/2N}}{\log^{1/2}(1+n)}\right\}$$

) is strongly summable to zero (see ([8]), pp. 270–282).

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* f_k is the k-th Fourier coefficient of the function $f(x)$.

As for the example showing that in estimate (2) the quantity $o\big(1/n^{1/2N}\big)$ cannot be replaced by $O\big(1/n^{1/2N+\varepsilon}\big)$, where $\varepsilon > 0$, for its construction one should consider the eigenfunctions of the N-dimensional ball and, in the ball internal with respect to it, a kernel of fractional order smoothly reduced to zero (see (11)) $K_{N/4+\varepsilon}(x_0, y)$, where x_0 is the center of the ball and y is the variable point.

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Note: Figure translations are in progress. See original paper for figures.