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TIKHONOV SPACES BY
MEANS OF
DIVERGENT NETS**

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Abstract

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MATHEMATICS

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A CHARACTERIZATION OF FUNCTIONALLY SEPARABLE AND TIKHONOV SPACES BY MEANS OF DIVERGENT NETS

(Presented by Academician A. N. Tikhonov, 4 VII 1969)

Let \mathcal{T}_1 be the class of all topological spaces satisfying the first separation axiom; \mathcal{T}_{fs} the class of all functionally separable spaces (i.e., topological spaces S having the property that for any two distinct points $s^0, s^1 \in S$ there exists a continuous real-valued function x on S for which $x(s^0) \neq x(s^1)$, or, equivalently, there exists a continuous function x on S such that $0 \leq x \leq 1$, $x(s^0) = 0$, $x(s^1) = 1$); \mathcal{T}_{cr} the class of all completely regular spaces (see ⁽¹⁾, p. 126); \mathcal{T}^n (respectively $\mathcal{T}^{n'}$) the class of all topological spaces S satisfying the following condition: for every divergent net $\{s_\alpha\}_{\alpha \in A}$ in S having at least n (respectively exactly n) limit points, there exists a bounded continuous real-valued function x , defined on S , such that the net $\{x(s_\alpha)\}_{\alpha \in A}$ diverges ($n = 1, 2$). Further, let

$$\mathcal{T}_1^1 = \mathcal{T}_1 \cap \mathcal{T}^1, \quad \mathcal{T}_1^2 = \mathcal{T}_1 \cap \mathcal{T}^2, \quad \mathcal{T}_1^{2'} = \mathcal{T}_1 \cap \mathcal{T}^{2'}, \quad \mathcal{T}_1^{1',2'} = \mathcal{T}_1 \cap \mathcal{T}^{1',2'}, \quad \mathcal{T}_{1,cr} = \mathcal{T}_1 \cap \mathcal{T}_{cr}.$$

Spaces of the class $\mathcal{T}_{1,cr}$ are called Tikhonov spaces (see ⁽¹⁾, p. 126).

In the present paper the following theorems are proved.

Theorem 1. $\mathcal{T}_{fs} = \mathcal{T}_1^2 = \mathcal{T}_1^{2'}$.

Theorem 2. $\mathcal{T}_{1,cr} = \mathcal{T}_1^1 = \mathcal{T}_1^{1',2'}$.

Theorem 3. Let $\{S_\beta\}_{\beta \in B}$ be a family of topological spaces, and let T be their product. Then

$$1^\circ. \quad T \in \mathcal{T}_1^2 \equiv \forall_{\beta \in B} S_\beta \in \mathcal{T}_1^2. \quad 2^\circ. \quad T \in \mathcal{T}_1^1 \equiv \forall_{\beta \in B} S_\beta \in \mathcal{T}_1^1.$$

These theorems adjoin the results of the article ⁽²⁾, in which it was proved that the class of all compact spaces coincides with the class $\mathcal{T}_1^0 = \mathcal{T}_1 \cap \mathcal{T}^0$, where \mathcal{T}^0 is the class of all topological spaces S having the following property: for every divergent net $\{s_\alpha\}_{\alpha \in A}$ in S there exists a bounded continuous real-valued function x , defined on S , such that the net $\{x(s_\alpha)\}_{\alpha \in A}$ diverges.

Proof of Theorem 1. Taking into account the obvious implication $S \in \mathcal{T}_1^2 \Rightarrow S \in \mathcal{T}_1^{2'}$, it is necessary to establish that: a) $S \in \mathcal{T}_{fs} \Rightarrow S \in \mathcal{T}_1^2$, and b) $S \in \mathcal{T}_1^{2'} \Rightarrow S \in \mathcal{T}_{fs}$.

- a) Suppose that $S \in \mathcal{T}_{fs}$ and that $\{s_\alpha\}_{\alpha \in A}$ is a net in S having at least 2 limit points. Let s^0 and s^1 be any two of them. Take a continuous function x on S such that the conditions $0 \leq x \leq 1$, $x(s^0) = 0$, $x(s^1) = 1$ are satisfied; put

$$V^0 = \{s : s \in S, x(s) \leq 1/3\}, \quad V^1 = \{s : s \in S, x(s) \geq 2/3\}, \quad A_i = \{\alpha : \alpha \in A, s_\alpha \in V^i\} \quad (i = 0, 1).$$

Since s^0 and s^1 are limit points of the net $\{s_\alpha\}_{\alpha \in A}$, and V^0 and V^1 are neighborhoods respectively of s^0 and s^1 , the sets A_0 and A_1 are cofinal with A . Therefore $\{x(s_\alpha)\}_{\alpha \in A_i}$ ($i = 0, 1$) are subnets of the net $\{x(s_\alpha)\}_{\alpha \in A}$. Hence it follows that the net $\{x(s_\alpha)\}_{\alpha \in A}$ diverges, for otherwise we would have had a common limit point of all its subnets, which in fact does not exist, since

$\limsup_{\alpha \in A_0} x(s_\alpha) \leq 1/3$, while $\liminf_{\alpha \in A_1} x(s_\alpha) \geq 2/3$. Thus it has been shown that $S \in \mathcal{T}_{fs} \Rightarrow S \in \mathcal{T}^2$. But $S \in \mathcal{T}_{fs} \Rightarrow S \in \mathcal{T}_1$; consequently, $S \in \mathcal{T}_{fs} \Rightarrow S \in \mathcal{T}_1^2$.

- b) Let $S \in \mathcal{T}_1^2$, $s_1, s_2 \in S$, $s_1 \neq s_2$. Consider the sequence $\{s_n\}_{n \in \mathbb{N}}$, where $s_n = s_1$ for odd n and $s_n = s_2$ for even n . This sequence has exactly two limit points (s_1 and s_2) and diverges (taking into account that $S \in \mathcal{T}_1$); hence, by the condition, there exists a bounded real-valued function x , continuous on S , such that the sequence $\{x(s_n)\}_{n \in \mathbb{N}}$ diverges. But this is possible only when $x(s_1) \neq x(s_2)$. Thus, $S \in \mathcal{T}_1^2 \Rightarrow S \in \mathcal{T}_{fs}$.

For the proof of Theorem 2 the following two lemmas will be needed.

Lemma 1. Let $S \in \mathcal{T}_{fs}$, let s^0 be a non-isolated point of the space S , and let $\{V_\alpha(s^0)\}_{\alpha \in A}$ be the system of all possible neighborhoods of the point s^0 having the form

$$\{s : s \in S, |x(s) - x(s^0)| \leq \varepsilon\},$$

where x is a bounded real-valued function continuous on S , $\varepsilon > 0$. Then the set A , partially ordered by the relation $\alpha \leq \alpha'$, meaning, by definition, that $V_\alpha(s^0) \supseteq V_{\alpha'}(s^0)$, is directed and has no maximal element.

Proof. Let $\alpha_1, \alpha_2 \in A$,

$$V_{\alpha_i}(s^0) = \{s : s \in S, |x_i(s) - x_i(s^0)| \leq \varepsilon_i\} \quad (i = 1, 2).$$

Putting

$$x_3(s) = \max(|x_1(s) - x_1(s^0)|, |x_2(s) - x_2(s^0)|), \quad \varepsilon_3 = \min(\varepsilon_1, \varepsilon_2),$$

$$V_{\alpha_3}(s^0) = \{s : s \in S, x_3(s) \leq \varepsilon_3\},$$

we shall have $\alpha_i \leq \alpha_3$ ($i = 1, 2$), which means that the set A is directed.

Let α' be an arbitrarily fixed element of A . Choose in the set $V_{\alpha'}(s^0)$ some point s^1 different from s^0 (such a point exists, since s^0 is a non-isolated point of the space S). By the condition, there exists a bounded function x_0 , continuous on S , such that $x_0(s^0) = 0$, $x_0(s^1) = 1$. Put

$$V_{\alpha''}(s^0) = \{s : s \in S, |x_0(s)| \leq 1/2\},$$

and let α''' be some majorant of the set $\{\alpha', \alpha''\}$; then $\alpha' \neq \alpha'''$, for

$$s^1 \in V_{\alpha'}(s^0) \setminus V_{\alpha''}(s^0) \subseteq V_{\alpha'}(s^0) \setminus V_{\alpha'''}(s^0).$$

Since α' was chosen arbitrarily, this shows that A has no maximal element.

Lemma 2. Let $B = A^n$, where A is a set directed by some relation \leq , in which there is no maximal element, and let n be an arbitrarily fixed natural number. Let B_1 be the diagonal in B , and let B_k ($k = 2, \dots, n$) be the set in B consisting of all possible rows (a_1, \dots, a_n) such that among the elements a_1, \dots, a_n there are exactly k distinct ones. If B is partially ordered by the relation

$$(a_1, \dots, a_n) \leq (a'_1, \dots, a'_n),$$

meaning, by definition, that $a_i \leq a'_i$ ($i = 1, \dots, n$), then each B_k ($k = 1, \dots, n$) will be cofinal with B .

Proof. Let $\beta = (a_1, \dots, a_n)$ be an arbitrarily chosen element of the set B ; let α'_1 be some majorant of the set $\{a_1, \dots, a_n\}$. Since $\beta_1 = (\alpha'_1, \dots, \alpha'_1) \in B_1$ and $\beta \leq \beta_1$, this proves that B_1 is cofinal with B . Let $2 \leq k \leq n$. Since A has no maximal element, there exist elements $a'_2, \dots, a'_k \in A$ such that $\alpha'_1, a'_2, \dots, a'_k$ are pairwise distinct and

$$\alpha'_1 \leq a'_2 \leq \dots \leq a'_k.$$

The element

$$\beta_k = (\alpha'_1, a'_2, \dots, a'_k, a'_k, \dots, a'_k) \in B_k$$

and $\beta \leq \beta_k$. Hence B_k is cofinal with B ($k = 2, \dots, n$).

Proof of Theorem 2. Taking into account the obvious implication

$$S \in \mathcal{T}_1^1 \Rightarrow S \in \mathcal{T}_1^{1',2'},$$

it is necessary to establish that: a) $S \in \mathcal{T}_{1,cr} \Rightarrow S \in \mathcal{T}_1^1$ and b) $S \in \mathcal{T}_1^{1',2'} \Rightarrow S \in \mathcal{T}_{1,cr}$.

- a) Suppose that $S \in \mathcal{T}_{cr}$ and that $\{s_\alpha\}_{\alpha \in A}$ is a divergent net in S having at least one limit point. Let s^0 be any one of them. Since s^0 is not a point of convergence of the net $\{s_\alpha\}_{\alpha \in A}$, there exists an open neighborhood V^0 of the point s^0 such that the set

$$A_0 = \{\alpha : \alpha \in A, s_\alpha \in S \setminus V^0\}$$

is cofinal with A . Choose a function x , continuous on S , so that the following conditions are satisfied:

$$0 \leq x \leq 1, \quad x(s^0) = 0, \quad x = 1 \text{ on } S \setminus V^0,$$

and put

$$V^1 = \{s : s \in S, x(s) \leq 1/2\}, \quad A_1 = \{\alpha : \alpha \in A, s_\alpha \in V^1\}.$$

Since s^0 is a limit point of the net $\{s_\alpha\}_{\alpha \in A}$ and V^1 is a neighborhood of the point s^0 , A_1 is cofinal with A . From this we conclude, as in the proof of item a) of Theorem 1, that the net $\{x(s_\alpha)\}_{\alpha \in A}$ diverges (since $\lim_{\alpha \in A_0} x(s_\alpha) = 1$, while $\limsup_{\alpha \in A_1} x(s_\alpha) \leq 1/2$). Thus,

$$S \in \mathcal{T}_{cr} \Rightarrow S \in \mathcal{T}^1,$$

whence

$$S \in \mathcal{T}_{1,cr} \Rightarrow S \in \mathcal{T}_1^1.$$

b) To prove the implication

$$S \in \mathcal{T}_1^{1',2'} \Rightarrow S \in \mathcal{T}_{1,cr},$$

or, equivalently, the implication

$$(S \in \mathcal{T}_{fs}) \wedge (S \notin \mathcal{T}_{cr}) \Rightarrow S \notin \mathcal{T}^1, \quad (*)$$

we first note the following two facts: 1) $S \in \mathcal{T}_{cr}$ if and only if, for every point $s^0 \in S$, the system $\{V_\alpha(s^0)\}_{\alpha \in A}$ (see Lemma 1) is a base of neighborhoods of the point s^0 ; 2) if $S \in \mathcal{T}_{fs}$ and s^0 is an isolated point of the space S , then the function x , defined on S by the equalities $x(s^0) = 0$, $x(s) = 1$ for $s \neq s^0$, is continuous on S , and therefore

$$\{s^0\} = \{s : s \in S, x(s) \leq 1/2\} \in \{V_\alpha(s^0)\}_{\alpha \in A}.$$

Proceeding to the proof of the implication (*), suppose that

$$(S \in \mathcal{T}_{fs}) \wedge (S \notin \mathcal{T}_{cr});$$

then (by virtue of the condition $S \notin \mathcal{T}_{cr}$) in S there is a point s^0 and a neighborhood V of this point such that

$$V_\alpha(s^0) \cap (S \setminus V) \neq \emptyset$$

for every $\alpha \in A$. Hence, and from the condition $S \in \mathcal{T}_{fs}$ (which entails $S \in \mathcal{T}_1$), it follows that s^0 is a non-isolated point of the space S . Put $B = A^2$ and partially order A and B as was done respectively in Lemma 1 and in Lemma 2 for $n = 2$. Then A will have no maximal element (Lemma 1), and each B_k ($k = 1, 2$) will

be a cofinal part of B (Lemma 2). We construct in S a net $\{s_\beta\}_{\beta \in B}$ as follows: if $\beta = (a, a) \in B_1$, then as s_β we take some element of $V_a(s^0) \cap (S \setminus V)$; if $\beta \in B_2$, then we put $s_\beta = s^0$. Since B_2 is cofinal in B , s^0 is a cluster point of the net $\{s_\beta\}_{\beta \in B}$. However, s^0 is not a point of convergence of the net $\{s_\beta\}_{\beta \in B}$, for V is a neighborhood of the point s^0 ,

$$B_1 = \{\beta : \beta \in B, s_\beta \notin V\},$$

and B_1 is cofinal in B . We shall show that any point s^1 distinct from s^0 cannot be a cluster point of the net $\{s_\beta\}_{\beta \in B}$. Indeed, the condition $S \in \mathcal{T}_{fs}$ means the existence of a continuous function x on S such that $x(s^0) = 0$, $x(s^1) = 1$. Putting

$$V^0 = \{s : s \in S, |x(s)| \leq 1/3\}, \quad V^1 = \{s : s \in S, |x(s)| \geq 2/3\},$$

we shall have $V^0 = V_{\alpha_0}(s^0)$ for some $\alpha_0 \in A$, and $V_{\alpha_0}(s^0) \cap V^1 = \emptyset$. It follows that $s_\beta \notin V^1$ for $\beta \geq \beta_0 = (\alpha_0, \alpha_0)$. This is precisely to say that s^1 is not a cluster point of the net $\{s_\beta\}_{\beta \in B}$ (since V^1 is a neighborhood of the point s^1). Thus, the net $\{s_\beta\}_{\beta \in B}$ diverges and has exactly one cluster point, s^0 . If we verify that for every continuous function x on S the net $\{x(s_\beta)\}_{\beta \in B}$ converges, then thereby the required result will be established:

$$S \notin \mathcal{T}^1.$$

We shall show that the net $\{x(s_\beta)\}_{\beta \in B}$ converges to $x(s^0)$. Choose a number $\varepsilon > 0$ and consider the neighborhood

$$V_{\alpha_\varepsilon}(s^0) = \{s : s \in S, |x(s) - x(s^0)| \leq \varepsilon\}$$

of the point s^0 . Obviously $s_\beta \in V_{\alpha_\varepsilon}(s^0)$ for

$$\beta \geq \beta_\varepsilon = (\alpha_\varepsilon, \alpha_\varepsilon),$$

and hence

$$|x(s_\beta) - x(s^0)| \leq \varepsilon$$

for $\beta \geq \beta_\varepsilon$. This proves, in view of the arbitrariness of ε , that

$$\lim_{\beta \in B} x(s_\beta) = x(s^0).$$

Proof of item 1° of Theorem 3. Taking Theorem 1 into account, it is necessary to show that

$$T \in \mathcal{T}_{fs} \equiv \forall_{\beta \in B} S_\beta \in \mathcal{T}_{fs}.$$

a) Let $T \in \mathcal{T}_{fs}$, $\beta_0 \in B$, $s_{\beta_0}^1, s_{\beta_0}^2 \in S_{\beta_0}$, $s_{\beta_0}^1 \neq s_{\beta_0}^2$. Choose in each S_β , for $\beta \neq \beta_0$, an element s_β^1 , and define in T elements t^1, t^2 , putting

$$t^1(\beta_0) = s_{\beta_0}^1, \quad t^2(\beta_0) = s_{\beta_0}^2, \quad t^1(\beta) = t^2(\beta) = s_\beta^1$$

for $\beta \neq \beta_0$. Since $t^1 \neq t^2$, by the condition there exists a function y , continuous on T , such that

$$y(t^1) \neq y(t^2).$$

Let

$$T_0 = \{t : t \in T, t(\beta) = s_\beta^1 \text{ for } \beta \neq \beta_0\},$$

and let $f : T_0 \rightarrow S_{\beta_0}$ be the mapping defined by the equality

$$f(t) = t(\beta_0)$$

for every $t \in T_0$. This is a one-to-one and bicontinuous mapping of T_0 onto S_{β_0} . Put

$$x(s_{\beta_0}) = y(f^{-1}(s_{\beta_0}))$$

for each $s_{\beta_0} \in S_{\beta_0}$. Obviously, the function x is continuous on S_{β_0} , and moreover

$$x(s_{\beta_0}^1) = y(f^{-1}(s_{\beta_0}^1)) = y(t^1) \neq y(t^2) = y(f^{-1}(s_{\beta_0}^2)) = x(s_{\beta_0}^2).$$

Since β_0 is an arbitrary element of B , the implication

$$T \in \mathcal{J}_{fs} \Rightarrow \forall_{\beta \in B} S_\beta \in \mathcal{J}_{fs}$$

is proved.

b) Let

$$\forall_{\beta \in B} S_\beta \in \mathcal{J}_{fs}, \quad t^1, t^2 \in T, \quad t^1 \neq t^2.$$

Then $t^1(\beta_0) \neq t^2(\beta_0)$ at least for one $\beta_0 \in B$ there exists a function x , continuous on S_{β_0} , such that $x(t^1(\beta_0)) \neq x(t^2(\beta_0))$. Put $y(t) = x(t(\beta_0))$ for any $t \in T$. It is obvious that the function y is continuous on T , and $y(t^1) \neq y(t^2)$. This proves the implication

$$\forall_{\beta \in B} S_\beta \in \mathcal{J}_{fs} \Rightarrow T \in \mathcal{J}_{fs}.$$

Proof of item 2° of Theorem 3 differs only insignificantly from the proof of the assertion

$$T \in \mathcal{J}^0 \equiv \forall_{\beta \in B} S_\beta \in \mathcal{J}^0,$$

given in article (2), and therefore it is omitted.

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² M. A. Goldman, *Latvian Math. Yearbook*, 8, 1970.

Note: Figure translations are in progress. See original paper for figures.

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