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Abstract

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MATHEMATICS

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PROPERTIES OF GREEN' S FUNCTIONS AND GENERALIZED SOLUTIONS OF ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Generalized solutions of such problems for which the system of boundary operators is normal have been well studied. Such a system makes it possible to construct the adjoint problem and to obtain estimates of generalized solutions by means of Green' s formula and estimates of classical solutions of the adjoint problem (see ([1–3])). Here Green' s functions will be used for the construction of generalized solutions. This makes it possible to consider generalized solutions of elliptic boundary-value problems without the assumption that the system of boundary operators is normal, and also to enlarge the class of admissible solutions and right-hand sides of the problem.

We consider the boundary-value problem:

$$\mathcal{L}u = \varphi_0(x) \quad \text{for } x \in \Omega; \qquad B_j u = \varphi_j(x) \quad \text{for } x \in S. \tag{1}$$

Here $j = 1, \dots, m$; Ω is a closed bounded domain of n-dimensional space ($n \geq 2$); $x = (x_1, \dots, x_n)$; S is the boundary of Ω; the function $\varphi_0(x)$ is defined in Ω, $\varphi_j(x)$ are defined on S; $\mathcal{L}$ is an elliptic operator of order $2m$, defined in Ω; B_j are differential operators of order $m_j \leq 2m - 1$, defined on S:

$$\mathcal{L} \equiv \sum_{0 \leq |\beta| \leq 2m} a_\beta(x) D_x^\beta, \qquad B_j \equiv \sum_{0 \leq |\beta| \leq m_j} b_{j\beta}(x) D_x^\beta \left(D_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \right). \tag{2}$$

Our assumptions concerning the operators $\mathcal{L}, B_j$ and the boundary S are as follows: $\mathcal{L}$ is properly elliptic; the B_j cover $\mathcal{L}$ ([1]); all coefficients $a_\beta(x) \in C^{l_1+1}(\Omega)$; $b_{j\beta}(x) \in C^{l_1+1}(S)$; the boundary $S \in C^{l_1+2m+1}$ in the sense of ([4]); l_1 is an integer subject to the condition $l_1 \geq \max(2l_0 - 1,\, 2m)$, where

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$$l_0 = \max_j (2m - m_j).$$

We introduce some notation. If $u(x)$ is a function defined in Ω, then by Lu we denote the vector-function $Lu = (\mathcal{L}u, B_1 u, \dots, B_m u)$; by φ we denote the vector-function $\varphi = (\varphi_0, \dots, \varphi_m)$, where $\varphi_0(x), \mathcal{L}u(x)$ are defined in Ω; $B_j u(x), \varphi_j(x)$ $(j \geq 1)$ are defined on S. We shall write problem (1) in the form $Lu = \varphi$. In addition, put:

$$(\varphi \cdot \psi) = \int_\Omega \varphi_0(x)\psi_0(x)\,dx + \sum_{j=1}^m \int_S \varphi_j(x)\psi_j(x)\,dx_S;$$

$$(u \cdot v)_\Omega = \int_\Omega u(x)v(x)\,dx; \qquad (u \cdot v)_S = \int_S u(x)v(x)\,dx_S;$$

$C^{k+\alpha}(\Omega), C^{k+\alpha}(S)$ are Hölder spaces, $|\ |^\Omega_{k+\alpha}, |\ |^S_{k+\alpha}$ are the norms in these spaces; $0 < \alpha < 1$; $k \geq 0$ is an integer (see, for example, ([4])).

It is known that elliptic problems have a finite number of solvability conditions. The following lemma shows that these conditions can be written with the aid of a certain system of smooth functions.

Lemma 1. There exists a certain number N of vector-functions

$$\bar{\alpha}_k = (\alpha_{0k}, \alpha_{1k}, \dots, \alpha_{mk}), \qquad k = 1, \dots, N,$$

$$\alpha_{0k}(x) \in C^{l_1+\alpha}(\Omega), \qquad \alpha_{jk}(x) \in C^{l_1-2m+m_j+1+\alpha}(S), \qquad j = 1, \dots, m,$$

such that the problem $Lu = \bar{\varphi}$, $u \in C^{2m+\alpha}(\Omega)$, is solvable if and only if the conditions

$$(\bar{\varphi} \cdot \bar{\alpha}_k) = 0, \qquad k = 1, \dots, N \tag{3}$$

are satisfied.

The proof of this lemma is based on the representation of any solution of the problem $Lu = \varphi$ in the form of the sum of the principal part and the remainder, obtained in ([7]).

Denote by $\bar{P}$ the projection operator onto the subspace generated by the vectors $\bar{\alpha}_k$, and by P the projection operator onto the subspace of solutions of the homogeneous problem $Lu = 0$,

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$$\bar{P}\bar{\varphi} = \sum_{k=1}^{N} (\bar{\varphi} \cdot \bar{\alpha}_k)\, \tilde{\bar{\alpha}}_k, \qquad Pu = \sum_{i=1}^{N'} (u \cdot \tilde{u}_i)_\Omega u_i. \tag{4}$$

Here $u_i \in C^{2m+l_1-l_0+\alpha}(\Omega)$ are solutions of the homogeneous problem $Lu_i = 0$, all u_i are orthonormal, and every solution $u(x) \in C^{2m+\alpha}(\Omega)$ of the problem $Lu = 0$ is a linear combination of the u_i; all $\bar{\alpha}_k$ are also assumed orthonormal, i.e.

$$(\bar{\alpha}_i \cdot \tilde{\bar{\alpha}}_k) = \delta_{ik}, \qquad (u_i \cdot \tilde{u}_k)_\Omega = \delta_{ik},$$

and $\bar{a}, \tilde{u}$ denote the quantities complex conjugate to a, u.

Definition. We shall call the vector-function

$$\mathcal{G}(x,y) = \{\mathcal{G}_0(x,y), \mathcal{G}_1(x,y), \dots, \mathcal{G}_m(x,y)\},$$

where $\mathcal{G}_0(x,y)$ is defined for $x, y \in \Omega,\ x \neq y$, and $\mathcal{G}_1(x,y), \dots, \mathcal{G}_m(x,y)$ are defined for $x \in \Omega,\ y \in S,\ x \neq y$, the **Green vector-function** of the problem $Lu = \bar{\varphi}$, if the function

$$u(x) = \int_\Omega \mathcal{G}_0(x,y)\varphi_0(y)\, dy + \sum_{j=1}^{m} \int_S \mathcal{G}_j(x,y)\varphi_j(y)\, dy_S \tag{5}$$

for arbitrary $\varphi_0 \in C^\alpha(\Omega)$, $\varphi_j \in C^{2m-m_j+\alpha}(S)$, belongs to $C^{2m+\alpha}(\Omega)$ and is a solution of the problem

$$Lu = \bar{\varphi} - \bar{P}\bar{\varphi}, \qquad Pu = 0.$$

The functions $\mathcal{G}_j(x,y)$, $j = 0, 1, \dots, m$, are assumed integrable in y for every x lying inside Ω.

Theorem 1. The Green vector-function of the problem $Lu = \bar{\varphi}$ exists and is unique.

The definition given here of the Green vector-function is a generalization of the definition given in ([6], [7]), where the problem $Lu = \varphi$ was assumed to be uniquely solvable. The construction of $\mathcal{G}(x,y)$ differs only slightly from the analogous construction in ([7]) with the aid of the principal part $\bar{\mathcal{G}}^{(r)}(x,y)$, which remains the same, and a smooth remainder. Thus it is easy to show that the functions $\mathcal{G}_j(x,y)$ have derivatives $D_x^s D_y^t \mathcal{G}_j(x,y)$ wherever they are defined, for $s \leq 2m + l_1 - 2l_0 + 1$, $t \leq l_1$ $(j = 0)$, $t \leq l_1 - 2m + m_j + 1$ $(j \geq 1)$. These derivatives satisfy the estimates given in ([7]) (see the corollary to Theorem 3.3) for the derivatives of Green’ s functions in the case of unique solvability of the problem $Lu = \varphi$, if in those estimates the constant M_0 is taken equal to

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$$M_0 = \sup |u|^{\Omega}_{2m+1} \quad \text{for} \quad |\mathcal{L}u|^{\Omega}_{\alpha} + \sum_{j=1}^{m} |B_j u|^{S}_{2m-m_j+\alpha} = 1, \tag{6}$$

where the supremum is taken over those $u \in C^{2m+\alpha}(\Omega)$ for which $Pu = 0$.

The following property of Green' s functions is derived on the basis of the study of their principal part, carried out in ([5], [7]).

Theorem 2. Let $y \in S$ be some point; let m' be the maximal order of the derivative in the direction of the normal to S that is contained in the operators B_j $(j = 1, \dots, m)$ in some neighborhood of y.

If $m' \leq 2m - 2$, then

$$D^k_y \mathcal{G}_0(x, y) = 0 \quad \text{for } 0 \leq k \leq 2m - m' - 2,$$

where x is an arbitrary point of Ω, $x \neq y$; D^k_y is any derivative of order k at the point y.

Corollary. Suppose the conditions of Theorem 2 are satisfied. Then the functions $\alpha_{0i}(y)$, $i = 1, \dots, N$, entering into the solvability conditions (3), satisfy the equality

$$D^k_y \alpha_{0i}(y) = 0, \quad \text{where } y \in S,\ 0 \leq k \leq 2m - m' - 2.$$

Let us refine the definition of the number m'. Suppose that in some neighborhood of the point $y \in S$ a local coordinate system $x'_1, \dots, x'_n$ is chosen, with $x'_n = 0$ the equation of S; let B'_j be the operator B_j written in the coordinate system $x'_1, \dots, x'_n$. By $m'(y)$ we denote the maximal order of the derivative with respect to x'_n which occurs in the operators B'_j $(j = 1, \dots, m)$ at the point y. Obviously, $m'(y)$ does not depend on the choice of the local coordinate system $x'_1, \dots, x'_n$. By m' in Theorem 2 is denoted $\max_z m'(z)$, where z belongs to the indicated neighborhood of y. In what follows m' will denote $\max_y m'(y)$, when $y \in S$ is any point.

We turn to the consideration of generalized solutions. For this purpose we introduce the necessary functional spaces. Define the norms:

$$|u|^{\Omega}_{-(k+\alpha)} = \sup_{\Phi} |(u \cdot \Phi)_{\Omega}| \quad \text{provided } |\Phi|^{\Omega}_{k+\alpha} = 1;$$

$$|v|^{S}_{-(k+\alpha)} = \sup_{F} |(v \cdot F)_{S}| \quad \text{provided } |F|^{S}_{k+\alpha} = 1;$$

$$|u|^{\Omega, r}_{-(k+\alpha)} = \sup_{\Phi'} |(u \cdot \Phi')_{\Omega}| \quad \text{provided } |\Phi'|^{\Omega}_{k+\alpha} = 1,$$

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where the supremum is taken over all (normalized) $\Phi \in C^{k+\alpha}(\Omega)$, $F \in C^{k+\alpha}(S)$, $\Phi' \in C^{k+\alpha}(\Omega)$; Φ' vanishes on S together with all derivatives up to order r inclusive, $r \leq k$. Denote by $C^{-(k+\alpha)}(\Omega)$, $C_r^{-(k+\alpha)}(\Omega)$ the closures of the set of infinitely differentiable functions defined in Ω, respectively in the norms $|\cdot|^{\Omega}_{-(k+\alpha)}$, $|\cdot|^{\Omega,r}_{-(k+\alpha)}$. Denote by $C^{-(k+\alpha)}(S)$ the closure of the set of functions belonging to $C^{2m+\alpha}(S)$ in the norm $|\cdot|^{S}_{-(k+\alpha)}$.

It is easy to see that $C^{-(k+\alpha)}(\Omega)$, $C_r^{-(k+\alpha)}(\Omega)$, $C^{-(k+\alpha)}(S)$ are complete linear normed spaces with the norms defined above. Such spaces and norms are usually called negative. In the present case the corresponding positive spaces are $C^{(k+\alpha)}(\Omega)$, $C_r^{k+\alpha}(\Omega)$, $C^{k+\alpha}(S)$, where $C_r^{k+\alpha}(\Omega)$ is the subspace of $C^{k+\alpha}(\Omega)$ consisting of functions that vanish on S together with all derivatives up to order r inclusive.

Let u be a function defined in Ω; by $\partial^j u/\partial\nu^j$ we denote the function defined on S and equal to the derivative of order j in the direction of the inner normal to S of u.

Introduce the norm $|u|^{\Omega}_{-(k+\alpha),t}$, $t \geq 0$ an integer, by setting

$$|u|^{\Omega}_{-(k+\alpha),t} = |u|^{\Omega}_{-(k+\alpha)} + \sum_{j=0}^{t} \left| \frac{\partial^j u}{\partial \nu^j} \right|^{S}_{-(k+j+1+\alpha)},$$

and define the space $\widetilde{C}^{-(k+\alpha),t}$ as the completion of the set of infinitely differentiable functions defined in Ω with respect to the norm $|\cdot|^{\Omega}_{-(k+\alpha),t}$. Obviously, $\widetilde{C}^{-(k+\alpha),t}$ is a complete linear normed space.

We introduce also the space of vector-functions $\bar{\varphi} = \{\varphi_0, \varphi_1, \dots, \varphi_m\}$, associated with the operator $L = \{\mathcal{L}, B_1, \dots, B_m\}$. We shall say that $\bar{\varphi} \in \bar{C}_L^{-(k+\alpha)}$ if $\varphi_0 \in C_r^{-(k+2m+\alpha)}(\Omega)$ for $2m-2-m' \geq 0$; $\varphi_0 \in C^{-(k+2m+\alpha)}(\Omega)$ for $m' = 2m-1$; $\varphi_j \in C^{-(k+m_j+1+\alpha)}(S)$ $(j = 1, \dots, m)$.

$$|\bar{\varphi}|^{L}_{-(k+\alpha)} = |\varphi_0| + \sum_{j=1}^{m} |\varphi_j|^{S}{}_{-(k+m_j+1+\alpha)},$$

$$|\varphi_0| = \begin{cases} |\varphi_0|^{\Omega,\,2m-2-m'}_{-(k+2m+\alpha)} & \text{for } 2m-2-m' \geq 0, \\ |\varphi_0|^{\Omega}_{-(k+2m+\alpha)} & \text{for } m' = 2m-1; \end{cases}$$

$$|\;|^{L}_{-(k+\alpha)} \text{ is the norm in the space } \bar{C}_L^{-(k+\alpha)}.$$

Definition. We shall call u a generalized solution of the problem $Lu = \varphi$, where $\bar{\varphi} \in \bar{C}_L^{-(k+\alpha)}$ $(k \leq l_1 - 2m)$, if $u \in C^{-(k+\alpha)}(\Omega)$ and there exists a sequence of functions $u_i \in C^{2m+\alpha}(\Omega)$ such that the relations

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$$\lim_{i\to\infty} |u - u_i|^{\Omega}_{-(k+\alpha)} = 0, \qquad \lim_{i\to\infty} |\bar{\varphi} - Lu_i|^{L}_{-(k+\alpha)} = 0$$

hold.

In other words, we call u a generalized solution if it can be obtained as the limit of classical solutions in the corresponding negative norms.

We note that the operators $\bar{P}, P$ (4) are easily defined by continuity on the negative spaces, if one takes into account the smoothness of the functions $a_{jk}(x)$ according to Lemma 1 and of the homogeneous solutions $u_i(x)$ of the problem $Lu = 0$, and also the consequence of Theorem 2. In this way we obtain that the operator $\bar{P}$ is a bounded operator in the space $\bar{C}_L^{-(k+\alpha)}$ for $k \leq l_1 - 2m$, and the operator P is a bounded operator in the space $C^{-(k+\alpha)}(\Omega)$ for $k \leq l_1 + 2m - l_0$.

Theorem 3. If $\bar{\varphi} \in \bar{C}_L^{-(k+\alpha)}$, $\bar{P}\bar{\varphi} = 0$, then there exists a unique generalized solution of the problem $Lu = \varphi$ such that $Pu = 0$. This generalized solution belongs to $C^{-(k+\alpha),m'}$, and the estimates

$$M^{-1}|\bar{\varphi}|^{L}_{-(k+\alpha)} \leq |u|^{\Omega}_{-(k+\alpha),m'} \leq M|\bar{\varphi}|^{L}_{-(k+\alpha)}$$

are valid.

Here M depends only on the coefficients of the operators $\mathcal{L}, B_j$, the boundary S, M_0 (6), and α; more precisely, M depends only on the constant c defined in (5), on M_0 (6), and on α.

The proof of this theorem is based on the representation of the Green functions as the sum of the principal part and a remainder and on the properties of the operators giving the principal part of the Green function, which were considered in (5, 7). Here one also uses the property of the Green functions formulated in Theorem 2.

It follows from Theorem 3 that the operator L, defined by continuity on the space $C^{-(k+\alpha),m'}$, realizes a one-to-one and continuous mapping of the subspace $\widetilde{C}^{-(k+\alpha),m'}$, containing those $u \in C^{-(k+\alpha),m'}$ for which $Pu = 0$, onto the subspace $\bar{C}_L^{-(k+\alpha)}$, containing those $\bar{\varphi} \in \bar{C}_L^{-(k+\alpha)}$ for which $\bar{P}\bar{\varphi} = 0$.

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