DYNAMIC PROBLEM FOR A PLANE WITH A CRACK

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Abstract

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DYNAMIC PROBLEM FOR A PLANE WITH A CRACK

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The problem of steady-state vibrations of an infinite plane made of an ideally brittle material with a crack of length $2l$ along the real axis is considered. It is assumed that a normal load $p(\theta)\cos\omega t$ is applied to the crack faces.

We introduce elliptic coordinates, in which the crack contour corresponds to the value $\rho = 0$ $(0 \leq \theta \leq 2\pi)$.

The equations of motion and the relations of elasticity theory for the amplitude values of stresses and displacements in the chosen coordinate system have the form

$$\frac{\partial}{\partial \rho}(H\sigma_\rho) + \frac{\partial}{\partial \theta}(H\sigma_{\rho\theta}) + \frac{\partial H}{\partial \theta}\sigma_{\rho\theta} - \frac{\partial H}{\partial \rho}\sigma_\theta = -H^2\gamma\omega^2 u_\rho,$$

$$\frac{\partial}{\partial \theta}(H\sigma_\theta) + \frac{\partial}{\partial \rho}(H\sigma_{\rho\theta}) + \frac{\partial H}{\partial \rho}\sigma_{\rho\theta} - \frac{\partial H}{\partial \theta}\sigma_\rho = -H^2\gamma\omega^2 u_\theta, \tag{1}$$

$$\sigma_\rho = 2\mu\left(\frac{1}{H}\frac{\partial u_\rho}{\partial \rho} + \frac{1}{H^2}\frac{\partial H}{\partial \theta}u_\theta\right) + \lambda\frac{1}{H^2}\left[\frac{\partial}{\partial \rho}(Hu_\rho) + \frac{\partial}{\partial \theta}(Hu_\theta)\right],$$

$$\sigma_{\rho\theta} = \mu\left[\frac{\partial}{\partial \rho}\left(\frac{u_\theta}{H}\right) + \frac{\partial}{\partial \theta}\left(\frac{u_\rho}{H}\right)\right],$$

$$\sigma_\theta = 2\mu\left(\frac{1}{H}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{H^2}\frac{\partial H}{\partial \rho}u_\rho\right) + \lambda\frac{1}{H^2}\left[\frac{\partial}{\partial \rho}(Hu_\rho) + \frac{\partial}{\partial \theta}(Hu_\theta)\right]. \tag{2}$$

Here $H^2 = \frac{1}{2}l^2(\operatorname{ch} 2\rho - \cos 2\theta)$; γ is the density of the material; ω is the circular frequency; λ, μ are Lamé constants.

Equations (1) are satisfied identically if the displacements and stresses are represented through two functions

$$u_\rho = \frac{1}{H}\frac{\partial\varphi}{\partial\rho} + \frac{1}{H}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = \frac{1}{H}\frac{\partial\varphi}{\partial\theta} - \frac{1}{H}\frac{\partial\psi}{\partial\rho},$$

where φ and ψ are solutions of the equations

$$\nabla^2\varphi + 2k_1(\operatorname{ch}2\rho - \cos 2\theta)\varphi = 0, \qquad \nabla^2\psi + 2k_2(\operatorname{ch}2\rho - \cos 2\theta)\psi = 0,$$

$$\nabla^2 = \partial^2/\partial\rho^2 + \partial^2/\partial\theta^2, \qquad k_1 = \gamma\omega^2 l^2/4(\lambda+2\mu), \qquad k_2 = \gamma\omega^2 l^2/4\mu. \quad (3)$$

With this representation, equations (2) are rewritten in the form

$$\frac{1}{2\mu}\sigma_{\rho\theta} = \frac{1}{H}\frac{\partial}{\partial\theta}\left(\frac{1}{H}\frac{\partial\varphi}{\partial\rho} + \frac{1}{H}\frac{\partial\psi}{\partial\theta}\right) - \frac{1}{H^3}\frac{\partial H}{\partial\rho}\left(\frac{\partial\varphi}{\partial\theta} - \frac{\partial\psi}{\partial\rho}\right) + 2k_2\psi,$$

$$\frac{1}{2\mu}\sigma_\rho = -\frac{1}{H}\frac{\partial}{\partial\theta}\left(\frac{1}{H}\frac{\partial\varphi}{\partial\theta} - \frac{1}{H}\frac{\partial\psi}{\partial\rho}\right) - \frac{1}{H^3}\frac{\partial H}{\partial\rho}\left(\frac{\partial\varphi}{\partial\rho} + \frac{\partial\psi}{\partial\theta}\right) - 2k_2\varphi,$$

$$\frac{1}{2\mu}\sigma_\theta = \frac{1}{H}\frac{\partial}{\partial\theta}\left(\frac{1}{H}\frac{\partial\varphi}{\partial\theta} - \frac{1}{H}\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{H^3}\frac{\partial H}{\partial\rho}\left(\frac{\partial\varphi}{\partial\rho} + \frac{\partial\psi}{\partial\theta}\right) - \frac{\lambda}{2\mu}\frac{\gamma\omega^2}{(\lambda+2\mu)}\varphi. \quad (4)$$

Separating variables in equations (3), we obtain ordinary differential equations whose solutions are Mathieu functions ([1]).

Using the symmetry of the stress state and the conditions at infinity, we represent the solutions of equations (3) in the form

$$\varphi(\rho,\theta) = \sum_{m=0}^{\infty} C_m \operatorname{Fey}_{2m}(\rho,k_1)\operatorname{ce}_{2m}(\theta,k_1),$$

$$\psi(\rho,\theta) = \sum_{m=0}^{\infty} D_m \operatorname{Gey}_{2m+2}(\rho,k_2)\operatorname{se}_{2m+2}(\theta,k_2). \quad (5)$$

Here C_m, D_m are constants; $\operatorname{ce}_{2m}(\theta,k_1)$, $\operatorname{se}_{2m+2}(\theta,k_2)$ are periodic Mathieu solutions; $\operatorname{Fey}_{2m}(\rho,k_1)$, $\operatorname{Gey}_{2m+2}(\rho,k_2)$ are the second solutions corresponding to the modified Mathieu equation (1), and in what follows representations of these functions for small values of k_1, k_2 will be used.

The conditions on the contour of the cut ($\rho = 0$) have the form

$$\sigma_{\rho\theta} = 0, \qquad \sigma_\rho = q(\theta). \quad (6)$$

Taking (4) into account, from (6) we obtain

$$\left(\frac{\partial\varphi}{\partial\rho}+\frac{\partial\psi}{\partial\theta}\right)_{\rho=0}=-2k_2\sin\theta\int_0^\theta \psi\big|_{\rho=0}\sin\theta\,d\theta+l\sin\theta\,\chi_1,$$

$$\left(\frac{\partial\varphi}{\partial\theta}-\frac{\partial\psi}{\partial\rho}\right)_{\rho=0}=-2k_2\sin\theta\int_{\pi/2}^\theta \varphi\big|_{\rho=0}\sin\theta\,d\theta-\frac{l^2}{2\mu}\sin\theta\int_{\pi/2}^\theta \sigma_0\big|_{\rho=0}\sin\theta\,d\theta+$$

$$+l\sin\theta+\chi_2. \qquad (7)$$

where $\chi_1=0,\ \chi_2=0$ by virtue of the obvious equalities

$$u_\rho\big|_{\rho=0,\ \theta=0}=0, \qquad u_\theta\big|_{\rho=0,\ \theta=\pi/2}=0.$$

We expand the function $q(\theta)$ specified on the contour of the cut in a series in Mathieu functions

$$q(\theta)=\sum_{m=0}^\infty q_m\,\mathrm{ce}_{2m}(\theta,k_1), \qquad q_m=\frac{1}{\pi}\int_0^{2\pi}q(\theta)\mathrm{ce}_{2m}(\theta,k_1)\,d\theta. \qquad (8)$$

Substituting the series (5), (8) into (7) and taking into account the expansions in Fourier series of the periodic solutions of Mathieu' s equation (1),

$$\mathrm{ce}_{2m}(\theta,k_1)=\sum_{r=0}^\infty A_{2r}^{(2m)}\cos 2r\theta, \qquad \mathrm{se}_{2m+2}(\theta,k_2)=\sum_{r=0}^\infty B_{2r+2}^{(2m+2)}\sin(2r+2)\theta,$$

we obtain two infinite systems for determining the unknown constants C_m, D_m

$$\sum_{m=0}^\infty\left\{C_m\,\mathrm{Fey}'_{2m}(0,k_1)A_{2r}^{(2m)}+D_m\,\mathrm{Gey}_{2m+2}(0,k_2)\left[2rB_{2r}^{(2m+2)}-\frac{2r}{4r^2-1}k_2B_{2r}^{(2m+2)}+\right.\right.$$

$$\left.\left.+\frac{k_2}{2(2r+1)}B_{2r+2}^{(2m+2)}+\frac{k_2}{2(2r-1)}B_{2r-2}^{(2m+2)}\right]\right\}=0, \qquad B_{-2}=B_0=0 \quad (r=0,1,2\ldots), \qquad (9)$$

$$\sum_{m=0}^\infty\left\{C_m\,\mathrm{Fey}_{2m}(0,k_1)\left[2rA_{2r}^{(2m)}-k_2\frac{2r}{(4r^2-1)}A_{2r}^{(2m)}+\right.\right.$$

$$+\frac{1}{2}k_2\frac{(1+\delta_r^{(1)})}{(2r-1)}A_{2r-2}^{(2m)}+\frac{1}{2}k_2\frac{1}{(2r+1)}A_{2r+2}^{(2m)}\Bigg]\Bigg\} =$$

$$= \frac{l^2}{4\mu}\sum_{m=0}^{\infty} q_m\left[\frac{2r}{(4r^2-1)}A_{2r}^{(2m)}-\frac{1}{2}\frac{(1+\delta_r^{(1)})}{(2r-1)}A_{2r-2}^{(2m)}-\frac{1}{2}\frac{1}{(2r+1)}A_{2r+2}^{(2m)}\right]$$

$$(r=0,1,2,...),\qquad \delta_r^{(1)}=1 \text{ for } r=1;\qquad \delta_r^{(1)}=0 \text{ for } r=0. \tag{10}$$

We use the known relations (1)

$$\mathrm{ce}_{2m}(0,k_1)=\sum_{r=0}^{\infty}A_{2r}^{(2m)},\qquad \mathrm{se}'_{2m+2}(0,k_2)=\sum_{r=0}^{\infty}(2r+2)B_{2r+2}^{(2m+2)}.$$

Adding all the equations of system (9), and analogously (10), we find, by virtue of the uniform continuity of the coefficients,

$$C_m\,\mathrm{Fey}_{2m}(0,k_1)\mathrm{ce}_{2m}(0,k_1)+D_m\,\mathrm{Gey}_{2m+2}(0,k_2)\mathrm{se}'_{2m+2}(0,k_2)=0, \tag{11}$$

$$C_m\,\mathrm{Fey}_{2m}(0,k_1)\left(\sum_{r=1}^{\infty}2rA_{2r}^{(2m)}+k_2A_0^{(2m)}\right)+$$
$$+D_m\,\mathrm{Gey}'_{2m+2}(0,k_2)\sum_{r=0}^{\infty}B_{2r+2}^{(2m+2)}=-\frac{l^2}{4\mu}q_mA_0^{(2m)}. \tag{11}$$

Equating to zero the determinant of the system (11), we obtain equations for determining the natural frequencies

$$\Delta_m\equiv\mathrm{Fey}'_{2m}(0,k_1)\,\mathrm{Gey}'_{2m+2}(0,k_2)\,\mathrm{ce}_{2m}(0,k_1)\sum_{r=0}^{\infty}B_{2r+2}^{(2m+2)}-$$
$$-\mathrm{Fey}_{2m}(0,k_1)\,\mathrm{Gey}_{2m+2}(0,k_2)\,\mathrm{se}'_{2m+2}(0,k_2)\left(\sum_{r=0}^{\infty}2rA_{2r}^{(2m)}+k_2A_0^{(2m)}\right)=0. \tag{12}$$

If the given frequency is such that $\Delta_m\neq 0$, then from (11) we determine the unknowns C_m, D_m

$$D_m=-C_m\frac{\mathrm{Fey}'_{2m}(0,k_1)\,\mathrm{ce}_{2m}(0,k_1)}{\mathrm{Gey}_{2m+2}(0,k_2)\,\mathrm{se}'_{2m+2}(0,k_2)},$$

$$C_m = -\frac{l^2}{4\mu}\, q_m \frac{A_0^{(2m)}\, \mathrm{Gey}_{2m+2}(0,k_2)\, \mathrm{se}'_{2m+2}(0,k_2)}{\Delta_m}. \tag{13}$$

Let us use the known relations (1)

$$\mathrm{Fey}'_{2m}(0,k_1) = \frac{2}{\pi}\frac{p_{2m}}{A_0^{(2m)}}\, \mathrm{ce}_{2m}\left(\frac{\pi}{2}, k_1\right),$$

$$\mathrm{Gey}_{2m+2}(0,k_2) = -\frac{2}{\pi}\frac{s_{2m+2}\, \mathrm{se}'_{2m+2}(\pi/2,k_2)}{k_2 B_2^{(2m+2)}},$$

$$p_{2m} = \frac{\mathrm{ce}_{2m}(0,k_1)\, \mathrm{ce}_{2m}(\pi/2,k_1)}{A_0^{(2m)}}, \qquad s_{2m+2} = \frac{\mathrm{se}'_{2m+2}(0,k_2)\, \mathrm{se}'_{2m+2}(\pi/2,k_2)}{k_2 B_2^{(2m+2)}}.$$

Then, taking this into account, from (12), (13)

$$\Delta_m = \frac{2}{\pi}\left[p_{2m}^2\, \mathrm{Gey}'_{2m+2}(0,k_2)\sum_{r=0}^{\infty} B_{2r+2}^{(2m+2)} + \right.$$
$$\left. + s_{2m+2}^2\, \mathrm{Fey}_{2m}(0,k_1)\left(\sum_{r=0}^{\infty} 2r A_{2r}^{(2m)} + k_2 A_0^{(2m)}\right)\right], \tag{14}$$

$$D_m = C_m \frac{p_{2m}^2}{s_{2m+2}^2}, \qquad C_m = -\frac{l^2}{2\pi\mu}\, q_m \frac{A_0^{(2m)} s_{2m+2}^2}{\Delta_m}. \tag{15}$$

Taking (4) into account, the normal stresses at the points of the real axis on the continuation of the cut are

$$\sigma_\theta\Big|_{\theta=0} = \frac{2\mu}{l^2}\frac{1}{\mathrm{sh}^3\rho}\left[\mathrm{sh}\,\rho\left(\frac{\partial^2\varphi}{\partial\theta^2} - \frac{\partial^2\psi}{\partial\rho\,\partial\theta}\right)_{\theta=0} + \mathrm{ch}\,\rho\left(\frac{\partial\varphi}{\partial\rho} + \frac{\partial\psi}{\partial\theta}\right)_{\theta=0}\right] - \frac{\lambda\gamma\omega^2}{\lambda+2\mu}\,\varphi\Big|_{\theta=0} =$$
$$= \frac{2\mu}{l^2}\frac{1}{\mathrm{sh}^3\rho}\sum_{m=0}^{\infty}\Omega_m(\rho) - \frac{\lambda\gamma\omega^2}{\lambda+2\mu}\,\varphi\Big|_{\theta=0}. \tag{16}$$

Here it is denoted that

$$\sum_{m=0}^{\infty}\Omega_m(\rho) = \mathrm{sh}\,\rho\left(\frac{\partial^2\varphi}{\partial\theta^2} - \frac{\partial^2\psi}{\partial\rho\,\partial\theta}\right)_{\theta=0} + \mathrm{ch}\,\rho\left(\frac{\partial\varphi}{\partial\rho} + \frac{\partial\psi}{\partial\theta}\right)_{\theta=0} =$$

Fig. 1

Figure 1: Fig. 1

$$= \sum_{m=0}^{\infty} C_m \left\{ \operatorname{sh}\rho \left[\operatorname{Fey}_{2m}(\rho,k_1)\,\operatorname{ce}''_{2m}(0,k_1) - \frac{p_{2m}^2}{s_{2m+2}^2}\,\operatorname{Gey}'_{2m+2}(\rho,k_2)\,\operatorname{se}'_{2m+2}(0,k_2) \right] + \right.$$
$$\left. + \operatorname{ch}\rho \left[\operatorname{Fey}'_{2m}(\rho,k_1)\,\operatorname{ce}_{2m}(0,k_1) + \frac{p_{2m}^2}{s_{2m+2}^2}\,\operatorname{Gey}_{2m+2}(\rho,k_2)\,\operatorname{se}'_{2m+2}(0,k_2) \right] \right\}. \tag{17}$$

To establish the dependence between the crack length and the applied load, we use Irwin' s relation [2]

$$\lim_{s\to 0} \sqrt{2\pi s\,\sigma_\theta}\Big|_{\theta=0} = K_c, \tag{18}$$

where

$$s = x\big|_{\theta=0} - l = l(\operatorname{ch}\rho - 1) = 2l\,\operatorname{sh}^2\frac{\rho}{2}$$

is the distance from the end of the crack, and K_c is the fracture toughness [2]. Taking (16) and (17) into account, we obtain

$$\lim_{s\to 0} \sqrt{2\pi s\,\sigma_\theta}\Big|_{\theta=0} = \lim_{\rho\to 0} \frac{\sqrt{2l}\,\mu}{l^2}\,\frac{1}{4}\sum_{m=0}^{\infty}\frac{\Omega_m(\rho)}{\operatorname{sh}^2\rho/2} = \frac{\mu\sqrt{2l}}{2l^2}\sum_{m=0}^{\infty}\Omega''_m(0).$$

Fig. 1

We use the known relations

$$\operatorname{Fey}''_{2m}(0,k_1) = (a_{2m} - 2k_1)\,\operatorname{Fey}_{2m}(0,k_1),$$

$$\operatorname{Gey}''_{2m+2}(0,k_2) = (b_{2m+2} - 2k_2)\,\operatorname{Gey}_{2m+2}(0,k_2);$$

then we obtain the final result

$$-\frac{2\mu}{\sqrt{\pi}\,l^{3/2}}\sum_{m=0}^{\infty} C_m(2k_1 - 2k_2 - a_{2m} + b_{2m+2})\,p_{2m}^2 = K_c, \tag{19}$$

where a_{2m}, b_{2m+2} are the eigenvalues of the Mathieu functions $\operatorname{ce}_{2m}(\theta k_1)$ and $\operatorname{se}_{2m+2}(\theta, k_2)$, respectively.

Fig. 2

Figure 2: Fig. 2

Fig. 2

Figures 1 and 2 show the curves of dependence (19) at different frequencies of oscillation of the external load for the case of concentrated forces and for the case of a uniform load applied to the crack faces (the dashed line corresponds to the static case $\omega = 0$). The constructed solution shows that the inertial effect reduces the magnitude of the breaking load for a given crack length.

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Note: Figure translations are in progress. See original paper for figures.