ON THE STRUCTURE OF THE SET OF POSITIVE DEVIATIONS OF MEROMORPHIC FUNCTIONS

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Abstract

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MATHEMATICS

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ON THE STRUCTURE OF THE SET OF POSITIVE DEVIATIONS OF MEROMORPHIC FUNCTIONS

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§ 1. In order to study deeper asymptotic properties of functions meromorphic in $|z| < R$, the papers ([1−3]) introduced the notion of the magnitude of deviation of a meromorphic function $f(z)$

$$\beta(a, f) = \lim_{r \to R} \frac{\ln^+ M(r, a, f)}{T(r, f)},$$

where

$$M(r, a, f) = \max_{|z|=r} \frac{1}{|f(z) - a|}, \quad a \neq \infty,$$

$$M(r, \infty, f) = \max_{|z|=r} |f(z)|,$$

and $T(r, f)$ is the Nevanlinna characteristic of the function $f(z)$ meromorphic in $|z| < R$.

The results obtained on the properties of the quantities of deviations of $f(z)$ show that the distribution of values of meromorphic functions can be studied using not only the notion of the defect at the point a in the sense of Nevanlinna (see ([4]))

$$\delta(a, f) = \lim_{r \to R} \frac{m(r, a, f)}{T(r, f)}$$

and the notion of the defect of $f(z)$ at the point a in the sense of Valiron

$$\Delta(a, f) = \overline{\lim}_{r \to R} \frac{m(r, a, f)}{T(r, f)},$$

but also the notion of the magnitude of deviation of $f(z)$ at the point a.

Let, for a function $f(z)$ meromorphic in $|z| < R$ and of lower order* λ (see ([4]), p. 267),

$$D_{R,\lambda}(f) = \{a : \delta(a, f) > 0\},$$

$$\Omega_{R,\lambda}(f) = \{a : \beta(a, f) > 0\},$$

$$V_{R,\lambda}(f) = \{a : \Delta(a, f) > 0\}.$$

The set $D_{R,\lambda}(f)$ is called the set of defective values of $f(z)$ in the sense of Nevanlinna; the set $\Omega_{R,\lambda}(f)$ is naturally called the set of positive deviations of $f(z)$, and the set $V_{R,\lambda}(f)$ is called the set of defective values of $f(z)$ in the sense of Valiron.

Fundamental investigations of the structure of the sets $D_{R,\lambda}(f)$ and $V_{R,\lambda}(f)$ were carried out in ([4–8]).

* Recall that the lower order of a meromorphic function in $|z| < R$ is called

$$\lambda = \lim_{r\to R} \frac{\ln T(r, f)}{\ln \frac{1}{R-r}}, \quad R < \infty, \qquad \lambda = \lim_{r\to\infty} \frac{\ln T(r, f)}{\ln r}, \quad R = \infty.$$

This work is devoted to the study of the structure of the set $\Omega_{R,\lambda}(f)$. The results obtained show that, despite possible substantial differences between the sets $\Omega_{R,\lambda}(f)$, $D_{R,\lambda}(f)$, and $V_{R,\lambda}(f)$, nevertheless in most cases the set $\Omega_{R,\lambda}(f)$, just like the sets $D_{R,\lambda}(f)$ and $V_{R,\lambda}(f)$, is an exceptional set for $f(z)$.

§ 2. **Main results of the work.**

Theorem 1 (see ([1],[2])). *For $\lambda < \infty$, the set $\Omega_{\infty,\lambda}(f)$ is at most countable, and*

$$D_{\infty,\lambda}(f) \subseteq \Omega_{\infty,\lambda}(f) \subseteq V_{\infty,\lambda}(f).$$

The following three theorems characterize the differences between the deviations of meromorphic functions $\beta(a, f)$ and the values of their Nevanlinna defects $\delta(a, f)$ and Valiron defects $\Delta(a, f)$.

Theorem 2. *The sets $\Omega_{\infty,\infty}(f)$ and $\Omega_{\infty,\infty}(f) \setminus V_{\infty,\infty}(f)$ may have the cardinality of the continuum.*

Theorem 3. *There exists a set C of the cardinality of the continuum and an entire function $G(z)$ of infinite lower order such that $\beta(a, G) = \infty$ for every $a \in C$.*

Theorem 4. *For every ρ, $0 \le \rho \le \infty$, there exists a set C of the cardinality of the continuum and a function $g_\rho(z)$, meromorphic for $|z| < 1$, of order ρ, such that $\beta(a, g_\rho) = \infty$ for every $a \in C$.*

Corollary 1. *For any $\rho \ge 0$, the set $\Omega_{1,\rho}(f)$ may have the cardinality of the continuum.*

Let us note that for $\rho = 0$ Theorem 4 follows from the consideration of functions of bounded type (see ([4]), p. 210).

Next, for a function $f(z)$ meromorphic in $|z| < R$, for $0 < \alpha \le 1$, put

$$\beta_\alpha(a, f) = \lim_{r \to R} \frac{\ln^+ M(r, a, f)}{T^\alpha(r, f)}$$

$$\Omega^{(\alpha)}_{R,\lambda}(f) = \{a : \beta_\alpha(a, f) > 0\}.$$

The following assertions are certain analogues of the well-known Ahlfors–Nevanlinna theorem (see ([5]), p. 17; ([4]), p. 281) on the structure of the set $V_{R,\lambda}(f)$. In what follows, by E we denote a bounded set in the a-plane having positive capacity $C(E)$, and by $\mu(a)$ the distribution of unit mass on E solving Robin' s problem for E (see ([4]), pp. 123, 135).

Theorem 5. *For a function $f(z)$ meromorphic at $z \ne \infty$, the following assertions are valid:*

1) *For every $\alpha > 1/2$*

$$\int_E \beta_\alpha(a, f)\, d\mu(a) = 0;$$

2) *if $E \subset \{|a| \le 1\}$, $C(E)$ is the capacity of the set E, and λ is the lower order of $f(z)$ $(f(0) = 1)$, then*

$$\int_E \beta_{1/2}(a, f)\, d\mu(a) \le K_1(1 + \lambda^2) \left[1 + \ln^+ \frac{1}{C(E)}\right].$$

Corollary 2. *For every $\alpha > 1/2$, the set $\Omega^{(\alpha)}_{\infty,\lambda}(f)$ has inner capacity zero $(\Omega^{(1)}_{\infty,\lambda}(f) = \Omega_{\infty,\lambda}(f),\ \lambda \ge 0)$.*

Theorem 6. *If $f(z)$ is meromorphic for $|z| < 1$ and has lower order λ, then for every $\alpha > 1/2 + 3/\lambda$ $(\lambda > 0)$*

$$\int_E \beta_\alpha(a, f)\, d\mu(a) = 0.$$

Corollary 3. If $\alpha > 1/2 + 3/\lambda$, then the set $\Omega^{(\alpha)}_{1,\lambda}(f)$ has inner capacity zero.

§ 3. Theorem 1 was proved in works ([1], [2]).

Theorems 2, 3, and 4 are proved by means of a modification of known constructions ([6]–[10]).

Theorems 5 and 6 are proved by means of a modification of a method previously used by the author ([1], [2]).

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Note: Figure translations are in progress. See original paper for figures.