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**Abstract**

**Full Text**

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**MATHEMATICS**

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## ON PERTURBATION OF THE CONTINUOUS SPECTRUM

*(Presented by Academician I. M. Vinogradov on 21 II 1969)*

This paper gives a generalization of the results obtained in <sup>(1)</sup> to the case of infinite-dimensional perturbations.

1. As the unperturbed operator we still take the operator  $S$  of multiplication by the independent variable in the Hilbert space  $H = L_2(-\infty, \infty)$ , with domain  $D(S)$  consisting of those  $f(x) \in H$  for which also  $xf(x) \in H$ , and  $Sf(x) = xf(x)$ . To describe the perturbed operator we shall need several definitions.

Let  $\varepsilon$  be some positive number. By  $\Phi = \Phi(\varepsilon)$  we denote the set of those  $\varphi \in H$  for which there exists a continuation  $\varphi(z)$  of the function  $\varphi(x)$ , holomorphic for  $|\operatorname{Im} z| < \varepsilon$  and satisfying the condition

$$\sup_{|y| \leq \varepsilon_1} \int_{-\infty}^{\infty} |\varphi(x + iy)|^2 dx < \infty, \quad \varepsilon_1 < \varepsilon. \quad (1)$$

Functions  $\varphi \in \Phi$  will be called **completely regular** elements of  $H$ .

Let  $G$  be some (auxiliary abstract) Hilbert space and  $A$  a linear continuous operator:  $H = D(A) \rightarrow G$ . If the range  $R(A^*)$  of the adjoint operator  $A^* : G = D(A^*) \rightarrow H$  consists of completely regular elements, i.e.

$$R(A^*) \subset \Phi, \quad (2)$$

then, as is not hard to prove, there exists a vector-valued function  $\alpha(z)$  with values in  $G$ , holomorphic for  $|\operatorname{Im} z| < \varepsilon$ , such that

$$Af = \int_{-\infty}^{\infty} f(x)\alpha(x) dx, \quad f \in H, \quad (3)$$

where the integral exists in the sense of weak convergence in  $G$ . If one assumes that

$$\int_{-\infty}^{\infty} \|\alpha(x)\|_G^2 dx < \infty, \quad (4)$$

then  $A$  will be a completely continuous operator  $H \rightarrow G$ . Such an operator (i.e. an operator satisfying conditions (2) and (4)) we call **completely regular**.

**Definition 1.** An operator  $V : H = D(V) \rightarrow H$  is called a **completely regular perturbation** if it admits a representation of the form  $V = A^*B$ , where  $A$  and  $B$  are completely regular operators  $H \rightarrow G$ .

In this paper the operator  $T \stackrel{\text{df}}{=} S + V$ ,  $D(T) \stackrel{\text{df}}{=} D(S)$ , is studied, where  $V$  is a completely regular perturbation\*. Since the norm of  $V$  is not assumed to be small, the operator  $T$ , generally speaking, is not similar to  $S$ . In addition to eigenvalues of the usual type, the operator  $T$  may have spectral singularities (see Sec. 2), which, in particular, may be eigenvalues with nonzero index. The eigenfunctions of the continuous spectrum of the operator  $T$  turn out to be  $\delta$ -functions perturbed by boundary values of certain Cauchy-type integrals (see Sec. 3). The Parseval equality corresponding to the operator  $T$  is written by means of regularization of a certain integral that diverges near the spectral singularities.

2. It is convenient to describe the spectral properties of the operator  $T$  in terms of the operator function  $K(\zeta) : G \rightarrow G$

$$K(\zeta) \stackrel{\text{df}}{=} 1 + BS_\zeta A^*, \quad \text{Im } \zeta \neq 0, \quad (5)$$

where  $S_\zeta \stackrel{\text{df}}{=} (S - \zeta)^{-1}$ . For example, it is not hard to see that a nonreal  $\lambda$  is an eigenvalue of the operator  $T$  if and only if the operator  $K(\lambda)$  is not invertible in  $G$ . If  $\text{Im } \zeta \neq 0$  and  $K(\zeta)^{-1}$  exists, then  $\zeta$  belongs to the resolvent set of the operator  $T$ , and

$$T_\zeta \stackrel{\text{df}}{=} (T - \zeta)^{-1} = S_\zeta - S_\zeta A^* K(\zeta)^{-1} B S_\zeta. \quad (6)$$

The set of eigenvalues of the operator  $T$  is finite, and to each eigenvalue there corresponds a finite-dimensional root subspace.

Formula (6) makes it possible to conclude that, for any  $\varphi, \psi \in \Phi$ , the function  $(T_\zeta \varphi, \psi)$  (of the variable  $\zeta$ ) has continuations  $(T_\zeta \varphi, \psi)_+$  and  $(T_\zeta \varphi, \psi)_-$  from the half-planes respectively  $\text{Im } \zeta > 0$  and  $\text{Im } \zeta < 0$ , analytic in the half-planes respectively  $\text{Im } \zeta > -\varepsilon$  and  $\text{Im } \zeta < +\varepsilon$ , and having no singularities other than poles.

**Definition 2.** The real poles of the functions  $(T_\zeta \varphi, \psi)_+$  and  $(T_\zeta \varphi, \psi)_-$  are called **spectral singularities** of the operator  $T$ .

The set of spectral singularities of the operator  $T$  is also finite and coincides with the set of spectral singularities of the operator  $T^*$ . In order that a real  $\sigma$  be an eigenvalue of the operator  $T$  or  $T^*$ , it is necessary (but not sufficient) that  $\sigma$  be a spectral singularity. In this case the dimension of the root subspace corresponding to  $T^*$ , generally speaking, is not equal to the dimension of the root subspace corresponding to  $T$ , so that, generally speaking,  $\sigma$  has nonzero index. The spectral singularities coincide with the real poles of the operator-valued functions  $K_+(\zeta)^{-1}$  and  $K_-(\zeta)^{-1}$ , where  $K_+(\zeta)$  and  $K_-(\zeta)$  are the analytic continuations of  $K(\zeta)$  from the upper and lower half-planes, respectively.

To each nonreal eigenvalue  $\lambda$  of the operator  $T$  we assign the eigenprojection

$$P_\lambda \stackrel{\text{df}}{=} -\frac{1}{2\pi i} \oint T_\zeta d\zeta, \quad (7)$$

where inside the contour of integration there are no singularities of the resolvent  $T_\zeta$  other than  $\lambda$ . To each spectral singularity  $\sigma$  we assign two "eigenprojections"  $P_{\sigma+}$  and  $P_{\sigma-}$ ,

$$(P_{\sigma\pm}\varphi, \psi) = -\frac{1}{2\pi i} \oint (T_\zeta\varphi, \psi)_\pm d\zeta, \quad \varphi, \psi \in \Phi, \quad (8)$$

\* Some of the results formulated below are also valid under more general assumptions on the perturbation  $V$ .

with an analogous condition on the contour of integration. We note that the "projectors"  $P_{\sigma\pm}$  act from  $\Phi$  into the set of semilinear functionals defined on  $\Phi$ . All the operators  $P_\lambda$ ,  $P_{\sigma+}$ , and  $P_{\sigma-}$  are finite-dimensional. It is useful to apply formula (6) in order to express  $P_\lambda$ ,  $P_{\sigma+}$ , and  $P_{\sigma-}$  in terms of the residues of  $K(\zeta)$ ,  $K_+(\zeta)$ , and  $K_-(\zeta)$  at  $\zeta = \lambda$  and  $\zeta = \sigma$ , respectively.

3. If the real  $\xi$  is not a spectral singularity, then  $\xi$  belongs to the continuous spectrum of the operator  $T$ . In order to describe the jump of the resolvent  $T_\xi$  on the continuous spectrum, we introduce the operators:

$$\mathfrak{A}\varphi(\xi) \stackrel{\text{df}}{=} \varphi(\xi) - B^*K_+(\bar{\xi})^{*-1}AS_\xi\varphi_-(\xi), \quad \varphi \in \Phi, \quad |\text{Im } \zeta| < \varepsilon, \quad (9)$$

$$\mathfrak{B}\varphi(\xi) \stackrel{\text{df}}{=} \varphi(\xi) - A^*K_-(\xi)^{-1}BS_\xi\varphi_-(\xi);$$

here  $AS_\xi\varphi_-$  and  $BS_\xi\varphi_-$  are continuations from the lower half-plane of the functions  $AS_\xi\varphi$  and  $BS_\xi\varphi$ , which exist and are holomorphic (in the metric of  $G$ ) if  $\varphi \in \Phi$ . It turns out that, for arbitrary  $\varphi, \psi \in \Phi$  and for all  $\xi$  from some neighborhood of the real axis, with the exception of spectral singularities,

$$(T_\xi\varphi, \psi)_+ - (T_\xi\varphi, \psi)_- = 2\pi i \mathfrak{B}\varphi(\xi)\overline{\mathfrak{A}\psi(\xi)}. \quad (10)$$

From the “jump” formula (10) it is easy to derive that

$$\mathfrak{A}T^*\varphi(\xi) = \xi\mathfrak{A}\varphi(\xi), \quad \mathfrak{B}T\varphi(\xi) = \xi\mathfrak{B}\varphi(\xi) \quad (11)$$

for all real  $\xi$  distinct from the spectral singularities, which gives grounds for regarding the operators  $\mathfrak{A}$  and  $\mathfrak{B}$  as, respectively, the  $T^*$ - and  $T$ -Fourier transforms on the continuous spectrum. To describe the continuity properties of these operators, put\*

$$\varkappa_{\pm}(\zeta) \stackrel{\text{df}}{=} \prod_{\sigma} \left( \frac{\zeta - \sigma}{\zeta - i\varepsilon} \right)^{n_{\sigma\pm}}, \quad (12)$$

where the product sign extends over all spectral singularities  $\sigma$ , and  $n_{\sigma\pm}$  denotes the order of  $\sigma$  as a pole of  $K_{\pm}(\zeta)^{-1}$ . Then the operators

$$\mathfrak{A}_0\varphi(\xi) \stackrel{\text{df}}{=} \overline{\varkappa_+(\xi)} \mathfrak{A}\varphi(\xi), \quad \mathfrak{B}_0\varphi(\xi) \stackrel{\text{df}}{=} \varkappa_-(\xi) \mathfrak{B}\varphi(\xi) \quad (13)$$

admit extension from  $\Phi$  to continuous operators  $H \rightarrow H$ . Moreover, if  $\mathcal{D}$  denotes the differentiation operator in  $H$ , then from  $f \in D(\mathcal{D}^j)$  it follows that  $\mathfrak{A}_0f, \mathfrak{B}_0f \in D(\mathcal{D}^j)$  and

$$\|\mathcal{D}^j\mathfrak{A}_0f\|, \|\mathcal{D}^j\mathfrak{B}_0f\| \leq C_j \left( \sum_{j'=1}^j \|\mathcal{D}^{j'}f\|^2 \right)^{1/2}, \quad j = 0, 1, 2, \dots, \quad (14)$$

where  $C_j$  are certain constants.

The estimates (14) (for  $j = 1$ ) allow one to speak of the values of the functions  $\mathfrak{A}_0f(\xi), \mathfrak{B}_0f(\xi)$  for  $f \in D(\mathcal{D})$  for all real  $\xi$ . Denoting them respectively by  $(f, a_{\xi})$  and  $(f, b_{\xi})$ , and taking (11) into account, we may regard the functionals  $a_{\xi}$  and  $b_{\xi}$  as (generalized) eigenfunctions of the continuous spectrum of the operators  $T$  and  $T^*$ , respectively.

4. In conclusion we describe one variant of Parseval’s equality for the operator  $T$ . Put

$$\int_{+} \frac{f(x) dx}{(x - \sigma)^{\nu}} \stackrel{\text{df}}{=} \lim_{\tau \rightarrow 0} \int_{-\infty}^{\infty} \frac{f(x) dx}{[x - (\sigma + i\tau)]^{\nu}} \quad (15)$$

\* Since the operator  $K_{\pm}(\zeta) - 1$  is nuclear, there exists a determinant of the operator  $K_{\pm}(\zeta)$ . Everything that follows remains valid if relation (12) is replaced by the relation

$$\varkappa_{\pm}(\zeta) \stackrel{\text{df}}{=} \det K_{\pm}(\zeta).$$

provided that the limit on the right exists. Let  $n = \max(n_{\sigma_+} + n_{\sigma_-})$ , where the maximum is taken over all spectral singularities  $\sigma$ , and the numbers  $n_{\sigma_{\pm}}$  are the same as in (12). Then, for all  $f, g \in D(\mathcal{D}^n)$ ,

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)} dx = \int_+ (f, b_{\xi})\overline{(g, a_{\xi})} \frac{d\xi}{\varkappa_+(\xi)\varkappa_-(\xi)} + \sum_{\sigma} (P_{\sigma}f, g) + \sum_{\lambda} (P_{\lambda}f, g). \quad (16)$$

Here  $a_{\xi}$  and  $b_{\xi}$  are eigenfunctions of the continuous spectrum of  $T$  and  $T^*$  (see item 3);  $P_{\lambda}$  and  $P_{\sigma}$  are the eigenprojections of eigenvalues and spectral singularities (see (7) and (8)); let us clarify that  $P_{\sigma}$  is extended by continuity, in the norm  $(\|f\|^2 + \dots + \|f^{(n)}\|^2)^{1/2}$  from  $\Phi$  to  $D(\mathcal{D}^n)$ , while  $\sum_{\lambda}$  ranges over all nonreal eigenvalues  $\lambda$ , and  $\sum_{\sigma}$  over all spectral singularities  $\sigma$ . As is seen from (12), each spectral singularity  $\sigma$  is a pole of order  $n_{\sigma_+} + n_{\sigma_-}$  of the “spectral” density function  $1/\varkappa_+(\sigma)\varkappa_-(\sigma)$ . In this connection, a regularized integral  $\int_+$  arises on the right-hand side of (16). We understand this integral, roughly speaking, in the sense of (15)\*. The existence of such a regularization is ensured by the estimates (14).

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## References

1. V. E. Lyantse, DAN, **182**, No. 5 (1968).

\* The corresponding partition of unity is used, making it possible to find the limit (15) for each spectral singularity separately.

*Note: Figure translations are in progress. See original paper for figures.*

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