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Abstract

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MATHEMATICS

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ON ONE NEW CLASS OF SPACES AND SOME PROBLEMS FROM THE THEORY OF SYMMETRIZABILITY

(Presented by Academician P. S. Aleksandrov on 5 XI 1968)

The work is connected with the problems of the article by A. V. Arhangel'skiĭ⁽¹⁾. Below it is assumed throughout that X and Y are T_1 -spaces.

§ 1. **Definition 1.** A set of pairs

$$\{P'_\alpha, P''_\alpha \mid \alpha \in A = \bigcup_{n=1}^{\infty} A_n\}, \quad P'_\alpha \subseteq P''_\alpha \subseteq X,$$

will be called a σ -biconservative pairbase in X , if: 1) for $x \in X$ and its neighborhood U_x there exists $\alpha \in A$ such that $x \in P'_\alpha \subseteq P''_\alpha \subseteq U_x$, and 2)

$$B \subseteq A_n \Rightarrow \left[\bigcup_{\alpha}^B P'_\alpha \right] \subseteq \bigcup_{\alpha}^B P''_\alpha \quad (n = 1, 2, \dots)$$

(cf. Sider's σ -cushioned pairbase in⁽²⁾). Spaces with a σ -biconservative pairbase will be called **pseudodevelopable**.

Theorem 1. *The following three conditions are equivalent:*

1°. *In X there exists a σ -biconservative pairbase.*

2°. *In X there exists a family $\{P_n(x) \mid x \in X, n = 1, 2, \dots\}$ of subsets such that:*

- a) $\bigcap_n P_n(x) = \{x\}$ and $P_{n+1}(x) \subseteq P_n(x)$ ($x \in X, n = 1, 2, \dots$);
- b) *for $x \in X$ and its neighborhood U_x there exists n such that $x \in P_n(x) \subseteq U_x$;*
- c) $E \subseteq X \Rightarrow [E] \subseteq \bigcap_n \left(\bigcup_{x \in E} P_n(x) \right)$.

3°. *In X , for every open set U , one can assign a sequence of closed sets $\{F_n(U)\}$ such that:*

- a) $\bigcup_n F_n(U) = U$;
- b) $(U \subseteq V) \Rightarrow (F_n(U) \subseteq F_n(V))$ ($n = 1, 2, \dots$).

For the proof of the implication $1^\circ \Rightarrow 2^\circ$ one must put

$$P_n(x) = \bigcap \left\{ P'_\alpha \mid x \in P'_\alpha, \alpha \in \bigcup_{i=1}^n A_i \right\}^*$$

for the implication $2^\circ \Rightarrow 3^\circ$, put

$$F_n(U) = [\{x \mid P_n(x) \subseteq U\}]$$

and, finally, for the proof of the implication $3^\circ \Rightarrow 1^\circ$, one must put

$$A_n = \{(U, n) \mid X \supseteq U = In + U\},$$

$$(P'_{(U,n)}, P''_{(U,n)}) = (F_n(U), U).$$

The following proposition follows immediately from Definition 1 and Theorem 1.

Proposition 1. *Spaces with a σ -conservative (in particular, with a σ -discrete) closed net, as well as developable spaces (in the sense of Borges ⁽³⁾), are pseudodevelopable spaces.*

The converse, generally speaking, is false. Below is given Example 1 of a completely regular strongly symmetrizable space without a σ -conservative net. There is also known an example of a strongly symmetrizable paracompactum which is not a developable space, constructed by Heath ⁽⁴⁾. By our Theorem 8 both spaces are pseudodevelopable spaces.

* The intersection of the empty system is not excluded.

Theorem 2. Let $\{X_i \mid i = 1, 2, \dots\}$ be a sequence of T_1 -spaces and let X be the space with the weakest topology for which certain given mappings $f_i : X \rightarrow X_i$ are continuous. Then, if all X_i are pseudocircular spaces, X is also a pseudocircular space.

If $\{P_n^i(x)\}$ is the family from item 2^o of Theorem 1 for X_i , then in X we put

$$P_n(x) = \bigcap_{i=1}^n f_i^{-1}(P_n^i(f_i(x))).$$

Corollary 1. The class of pseudocircular spaces is hereditary, countably multiplicative, and is preserved under passage to the supremum of a countable number of topologies.

Theorem 3. Let $f : X \rightarrow Y$ be a closed continuous mapping. If X is a pseudocircular space, then Y is also a pseudocircular space.

If $\{F_n^X(U)\}$ is the family from item 3⁰ of Theorem 1 for X , then for a V -open set in Y we put

$$F_n^Y(V) = f(F_n^X(f^{-1}(V))).$$

Let us note that an open continuous image of a pseudocircular space, generally speaking, need not be a pseudocircular space. Indeed, the space “two circles” of P. S. Aleksandrov and P. S. Urysohn ⁽⁵⁾, example A_2 , is a nonmetrizable bicomactum with the first axiom of countability. As V. I. Ponomarev ⁽⁶⁾ showed, all spaces with the first axiom of countability are open continuous images of metric (and, consequently, pseudocircular) spaces. But by virtue of Theorem 10 given below, nonmetrizable bicomacta are not pseudocircular spaces.

Theorem 4. Let $X = \bigcup_{n=1}^{\infty} X_n$, where each X_n is closed in X and is a pseudocircular (sub)space; then X is a pseudocircular space. If $X = \bigcup_{\lambda} X_{\lambda}$, where each X_{λ} is closed and is a pseudocircular (sub)space, and the topology of X is weak with respect to the covering $\{X_{\lambda}\}$ ⁽⁷⁾, p. 232*, then X is a pseudocircular space.

Theorems 5 and 6 are a substantial generalization of results of A. V. Arhangel'skii for spaces with a σ -discrete closed network ⁽¹⁾, Theorem 2.8, and of J. Ceder for circular spaces ⁽²⁾, Theorem 6.2.

Theorem 5. A pseudocircular space with the weak first axiom of countability (see ⁽¹⁾, Definition 2.3) is symmetrizable (in the sense of A. V. Arhangel'skii, *ibid.*, p. 144).

In general, the converse is false. Namely, there exists a completely regular symmetrizable space that is not a pseudocircular space (see below, Example 2). The situation changes in the class of spaces with the first axiom of countability.

Theorem 6. In order that a space be strongly symmetrizable, it is necessary and sufficient that it be pseudocircular and satisfy the first axiom of countability.

Proposition 2. A Hausdorff pseudocircular space can be densely embedded in a symmetrizable space.

The following two theorems are generalizations of McAuley's results for strongly symmetrizable spaces ^(8, 9).

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* The latter, for example, holds if $\{X_{\lambda}\}$ is locally finite in X ; in particular, if X is the topological sum of $\{X_{\lambda}\}$ in the sense of Bourbaki.

Theorem 7. Into every system γ of open sets of a pseudo-circular space X one can inscribe a σ -discrete in X system of closed sets covering $\bigcup \gamma$.

Theorem 8. A collectively normal pseudo-circular space is hereditarily paracompact and condenses onto a metric space.

Theorem 9. The following properties are equivalent for a pseudo-circular space:

- 1°. Hereditary final compactness.
- 2°. Final compactness.
- 3°. The hereditary Suslin property (⁽¹⁾, p. 162).
- 4°. Every discrete system of sets of the space is at most countable.
- 5°. Hereditary separability.

In (¹⁰) S. Nedev proved the implication $2^\circ \Rightarrow 5^\circ$ for strongly symmetrizable spaces, and also the implications $5^\circ \Rightarrow 2^\circ \Rightarrow 1^\circ \Leftrightarrow 4^\circ$ for arbitrary symmetrizable spaces.

From Theorem 9 and Theorem 8 it follows that

Theorem 10. A Hausdorff countably compact pseudo-circular space is bicom-pact and metrizable.

Example 1 of a completely regular strongly symmetrizable space without a σ -conservative network. The set of points of the space is the set of points of the plane, and a base at an arbitrary point x is formed by the sets $U_n(x)$, equal to the unions of the one-point set $\{x\}$ and two open disks of radius $1/n$ tangent at x to the horizontal line.

Example 2 of a completely regular symmetrizable non-pseudo-circular space. The set of points of the space, as in the preceding example, is the set of points of the plane, and the balls of a certain symmetric in it are the sets

$$O_{1/n}(x) = U_n(x) \cup \left(\bigcup_{i>n} \left(\tilde{U}_{2^{i+n}} \left(x + \frac{1}{2^i} \right) \cup \tilde{U}_{2^{i+n}} \left(x - \frac{1}{2^i} \right) \right) \right),$$

where $U_n(x)$ is the same as in the preceding example, and $\tilde{U}_m(x)$ is equal to the union of the one-point set $\{x\}$ and two open disks of radius $1/m$ tangent at x to the vertical line.

§ 2. The following example solves E. Michael's problem on the symmetrizable-ity of the square of a symmetrizable space.

Example 3 of a countably regular symmetrizable space which at a single point does not satisfy the first axiom of countability, whose square is not a k -space and therefore is not symmetrizable.

Let Q_1 be the set of rational numbers $\neq 0$, $Q_2 = \{r\sqrt{2} \mid r \in Q_1\}$, $Q = Q_1 \cup Q_2 \cup \{0\}$, and $M = \{x + iy \mid x, y \in Q\}$. Put, for $n = 1, 2, \dots$,

$$R_n(0) = \{y \in M \mid |y| < 1/n\} \cap Q_1,$$

$$R_n(x) = \{y \in M \mid |x - y| < 1/n\} \quad \text{for } x \in Q_1,$$

$$R_n(x) = \{x\} \quad \text{for } x \in M - (Q_1 \cup \{0\}).$$

$\{R_n(x)\}$ is a weak base in the sense of A. V. Arhangel'skii (⁽¹⁾, p. 148) of a certain regular space X , symmetrizable by Theorem 5 and Proposition 1. It can be shown that X is not a k -space.

§ 3. A. V. Arhangel'skii showed in (⁽¹⁾) that on every quotient Π -image of a metric space (p. 247) there exists a symmetric with the weak Cauchy condition (p. 250). It turns out that the following holds.

Theorem 11. If X is a space with a symmetric satisfying the weak Cauchy condition, then X is a quotient Π -image of some metric space.

In the proof of Theorem 11 the following is used.

Proposition 3. If d is a symmetric with the weak Cauchy condition on X , then from every convergent sequence in X one can extract a fundamental (with respect to d) subsequence.

Not every symmetrizable space is a quotient Π -image of some metric space, as is shown by the following

Example 4 of a Hausdorff symmetrizable space without a symmetric satisfying the weak Cauchy condition. The set M of points is the set of points of the real line. The symmetric d

$$d(x, y) = \begin{cases} 1, & \text{if } x, y \text{ are irrational, } x \neq y, \\ |x - y|, & \text{otherwise} \end{cases}$$

defines on M the required space.

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