TWO-SIDED VARIATIONAL ESTIMATES

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Fig. 1

Figure 1: Fig. 1

Abstract

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MATHEMATICAL PHYSICS

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TWO-SIDED VARIATIONAL ESTIMATES

FOR THE POLARIZABILITY COEFFICIENTS

IN THE THEORY OF DIFFRACTION BY SMALL APERTURES

(Presented by Academician M. A. Leontovich on 27 III 1969)

Consider an infinite plane perfectly conducting screen of zero thickness with an aperture of arbitrary shape (Fig. 1). If $l \ll \lambda$ (l is the maximum linear dimension of the aperture, λ is the wavelength), then in the half-space $z > 0$, for $r \gg l$, the diffraction field may be ([1]) represented as the sum of the fields of electric and magnetic dipoles located at the screen with the metallized aperture and having moments

$$\vec{\mathcal{P}} = \mathbf{a}_z p \varepsilon_0 \mathcal{E}_z^0, \qquad \vec{\mathcal{M}} = \overleftrightarrow{\mathbf{m}} \cdot \vec{\mathcal{H}}_\tau^0,$$

$$\mathcal{E}_z^0 = \mathbf{a}_z \cdot \vec{\mathcal{E}}^0, \qquad \vec{\mathcal{H}}_\tau^0 = -\mathbf{a}_z \times [\mathbf{a}_z \times \vec{\mathcal{H}}^0], \tag{1}$$

where $\vec{\mathcal{E}}^0, \vec{\mathcal{H}}^0$ are the electric and magnetic fields of the exciting wave at the center of the metallized aperture, ε_0 is the dielectric permittivity of the medium in the SI system, p is the coefficient of electric polarizability, and $\overleftrightarrow{\mathbf{m}}$ is the tensor of magnetic polarizability of the aperture.

Fig. 1

Explicit expressions for these coefficients have been obtained ([2]) only for an elliptical aperture, while for some apertures of more complicated shape they have been found experimentally ([3]), since their theoretical determination requires the solution of rather complicated electrostatic and magnetostatic problems.

Therefore it is of interest to formulate variational principles that give two-sided estimates for the indicated quantities.

1. The coefficient of electric polarizability p is related to the solution of the following electrostatic problem. In the plane $z = 0$ there is a perfectly conducting screen, on one side of which (for $z < 0$) there exists a homogeneous electric field

$$\mathbf{E}^0 = \mathbf{a}_z \lim_{\lambda\to\infty} \mathcal{E}_z^0.$$

When an aperture is cut in it, a tangential electric field arises,

$$\mathbf{E}_\tau(\mathbf{x}) = -\nabla\varphi(\mathbf{x}),$$

and electric charge flows onto the shadow side of the screen with surface density

$$\sigma(\mathbf{x}) = -\varepsilon_0 \lim_{z\to+0} \frac{\partial\varphi(\mathbf{r})}{\partial z};$$

here φ is the electrostatic potential, and

$$\nabla = \mathbf{a}_1 \frac{\partial}{\partial x_1} + \mathbf{a}_2 \frac{\partial}{\partial x_2}.$$

To obtain the first variational principle, write the integral equation for $\varphi(\mathbf{x})$ on the aperture

$$\int_A ds'\, \nabla g(\mathbf{x},\mathbf{x}') \cdot \nabla'\varphi(\mathbf{x}') = -\pi E^0, \qquad \mathbf{x} \in A, \tag{2}$$

where $g = |\mathbf{x} - \mathbf{x}'|^{-1}$, the integral is understood in the sense of the principal value, and the index A indicates that the integration is carried out over the area of the aperture.

The leading term of the asymptotics of the potential in the half-space $z > 0$ has the form

$$\varphi(\mathbf{r}) = \frac{\mathbf{a}_z \cdot \mathbf{r}}{2\pi r^3} \int_A ds\, \varphi(\mathbf{x}), \qquad r \gg l. \tag{3}$$

Identifying (3) with the potential of an electric dipole located at the metallized aperture and oriented along $\mathbf{a}_z$, we find

$$p = \frac{1}{E^0} \int_A ds\, \varphi(\mathbf{x}). \tag{4}$$

Using the integral equation (2), by means of the standard procedure ([4]) we obtain for the quantity p the first homogeneous functional

$$p\{\varphi\} = \pi \frac{\left[\int_A ds\, \varphi(\mathbf{x})\right]^2}{\int_A ds \int_A ds'\, \nabla\varphi(\mathbf{x}) \cdot \vec{I}\, g(\mathbf{x}, \mathbf{x}') \cdot \nabla'\varphi(\mathbf{x}')}, \tag{5}$$

which is stationary on the class of functions that vanish on the contour of the aperture; $\vec{I}$ is the unit tensor. The exact solution $\varphi^0(\mathbf{x})$ of the integral equation (2) belongs to this class and, when approaching the contour of the aperture, satisfies the condition ([5]) $\varphi^0 \sim \xi^{1/2}$, where ξ is the distance to the contour. It is important to emphasize that the functional (5) attains a maximum on the true solution, for for the quantity $g(\mathbf{x}, \mathbf{x}')$ there exist expansions of the form

$$g(\mathbf{x}, \mathbf{x}') = \sum_n \int d\xi\, \Phi_n(\xi \mid \mathbf{x})\Phi_n(\xi \mid \mathbf{x}'), \tag{6}$$

where $\Phi_n(\xi \mid \mathbf{x})$ are certain real functions.

To obtain the second variational principle, we write the integral equation for $\sigma(\mathbf{x})$

$$\int_M ds'\, g(\mathbf{x}, \mathbf{x}')\sigma(\mathbf{x}') = -\frac{E^0}{2}\varepsilon_0 f(\mathbf{x}), \qquad f(\mathbf{x}) = \int_A ds'\, g(\mathbf{x}, \mathbf{x}'), \qquad \mathbf{x} \in M. \tag{7}$$

Here the index M means that the integration is carried out over the surface of the metal on the shadow side of the screen. With the aid of Green' s theorem we find the relation between the potential on the aperture and its normal derivative on the plane $z = +0$:

$$\varphi(\mathbf{x}) = \frac{1}{2\pi}\left\{\frac{1}{\varepsilon_0}\int_M ds'\, g(\mathbf{x}, \mathbf{x}')\sigma(\mathbf{x}') + \frac{1}{2}E^0 \int_A ds'\, g(\mathbf{x}, \mathbf{x}')\right\}, \qquad \mathbf{x} \in A. \tag{8}$$

Taking (4) into account, we obtain

$$p^* - p = -\frac{1}{2\pi\varepsilon_0 E^0}\int_M ds\, f(\mathbf{x})\sigma(\mathbf{x}), \tag{9}$$

where

$$p^* = \frac{1}{4\pi}\int_A ds \int_A ds'\, g(\mathbf{x}, \mathbf{x}') \tag{10}$$

in fact determines that part of the polarizability coefficient which is connected with the Kirchhoff approximation. Using (7) and (9), it is easy to obtain the second homogeneous functional

$$p^* - p\{\sigma\} = \frac{1}{4\pi} \frac{\left[\int_M ds\, f(\mathbf{x})\sigma(\mathbf{x})\right]^2}{\int_M ds \int_M ds'\, \sigma(\mathbf{x}) g(\mathbf{x}, \mathbf{x}') \sigma(\mathbf{x}')}, \tag{11}$$

stationary on the class of functions ensuring convergence of the integrals. The exact solution $\sigma^0(\mathbf{x})$ of the integral equation (7) decreases as $x \to \infty$

as x^{-3}, while on approaching the contour of the aperture from the metal side $\sigma \sim \xi^{-1/2}$. The indicated properties of the function $g(\mathbf{x}, \mathbf{x}')$ ensure the maximality of the functional $p^* - p\{\sigma\}$ on the true solution $\sigma^0(\mathbf{x})$, and hence the **minimality** of the desired quantity $p\{\sigma\}$.

Thus, the two constructed functionals always make it possible, for a given choice of trial functions, to indicate an upper and a lower bound for the coefficient of electric polarizability

$$p\{\varphi\} \leqslant p\{\varphi^0\} = p\{\sigma^0\} \leqslant p\{\sigma\}. \tag{12}$$

In conclusion we note that the computation of the functionals $p\{\varphi\}$ and $p\{\sigma\}$ can be carried out on the basis of only one trial function, since φ and σ are interrelated with the aid of Green' s theorem.

2. The tensor of magnetic polarizability $\overleftrightarrow{\mathbf{m}}$ is connected with the solution of the following magnetostatic problem. In the plane $z = 0$ there is situated an ideally conducting screen, on one side of which (for $z < 0$) there exists a homogeneous magnetic field $\mathbf{H}^0 = \lim_{\lambda\to\infty} \mathcal{H}^0_\lambda$. When an aperture is cut in it, there arises on it a normal component of the magnetic field $H_z(\mathbf{x}) = \lim_{z\to 0} \mathbf{a}_z \cdot \nabla \times \mathbf{A}$, and a surface electric current $\mathbf{K}(\mathbf{x}) = \lim_{z\to +0}[\mathbf{a}_z \times (\nabla \times \mathbf{A})]$ flows onto the shadow side of the screen; here $\mathbf{A}(\mathbf{r})$ is the vector potential.

To obtain the first variational principle, write the integral equation for $\mathbf{j}(\mathbf{x}) = \mathbf{a}_z \times \mathbf{A}(\mathbf{x})$

$$\int_A ds'\, \nabla g(\mathbf{x}, \mathbf{x}')\, \nabla' \cdot \mathbf{j}(\mathbf{x}') = \pi \mathbf{H}^0, \qquad \mathbf{x} \in A. \tag{13}$$

The first nonvanishing term of the multipole expansion of $\mathbf{A}(\mathbf{r})$ in the half-space $z > 0$ is the potential of a magnetic dipole

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2\pi}\nabla_r \frac{1}{r} \times \int_A ds'\, \mathbf{j}(\mathbf{x}'), \qquad r \gg l. \tag{14}$$

This dipole has the moment

$$\mathbf{M} \equiv \overleftrightarrow{\mathbf{m}} \cdot \mathbf{H}^0 = \int_A ds\, \mathbf{j}(\mathbf{x}), \tag{15}$$

is located at the metallized aperture and is oriented in the plane of the latter.

From the linearity of the problem it follows that $\mathbf{j}(\mathbf{x})$ can be represented in the form

$$\mathbf{j}(\mathbf{x}) = \mathbf{j}_1(\mathbf{x}) + \mathbf{j}_2(\mathbf{x}), \tag{16}$$

where the functions $\mathbf{j}_\alpha$ ($\alpha = 1, 2$), associated with the chosen rectangular coordinate system x_1, x_2 (Fig. 1), satisfy the equations

$$\int_A ds'\, \nabla g(\mathbf{x}, \mathbf{x}')\, \nabla' \cdot \mathbf{j}_\alpha(\mathbf{x}') = \pi \mathbf{a}_\alpha H^0_\alpha, \qquad H^0_\alpha = \mathbf{a}_\alpha \cdot \mathbf{H}^0 \tag{17}$$

and the condition at the edge: $\mathbf{n}(\mathbf{x}) \cdot \mathbf{j}(\mathbf{x}) \sim \xi^{1/2}$, $\mathbf{t}(\mathbf{x}) \cdot \mathbf{j}(\mathbf{x}) \sim \xi^{3/2}$; $\mathbf{n}$ and $\mathbf{t}$ are unit vectors normal and tangent to the contour of the aperture.

Taking (15)–(17) into account, we find the components of the symmetric tensor $\overleftrightarrow{\mathbf{m}}$ in the basis $\mathbf{a}_\alpha$

$$m_{\alpha\beta} = \frac{1}{H^0_\beta} \int_A ds\, \mathbf{a}_\alpha \cdot \mathbf{j}_\beta(\mathbf{x}), \qquad \alpha, \beta = 1, 2, \tag{18}$$

as well as their stationary representations

$$m_{\alpha\beta}\{\mathbf{j}_\alpha, \mathbf{j}_\beta\} = -\pi \frac{\int_A ds\, \mathbf{a}_\alpha \cdot \mathbf{j}_\beta(\mathbf{x}) \int_A ds\, \mathbf{a}_\beta \cdot \mathbf{j}_\alpha(\mathbf{x})}{\int_A ds \int_A ds'\, \nabla \cdot \mathbf{j}_\alpha(\mathbf{x})\, g(\mathbf{x}, \mathbf{x}')\, \nabla' \cdot \mathbf{j}_\beta(\mathbf{x}')}. \tag{19}$$

The functionals corresponding to the diagonal elements, upon substitution of the exact solutions $\mathbf{j}^0_\alpha$, have a **minimum**, which cannot be asserted for the off-diagonal elements.

Starting from the integral equation for the tangential magnetic field on the shadow side of the screen,

$$\int_M ds'\, g(\mathbf{x},\mathbf{x}')\,\mathbf{h}(\mathbf{x}') = -\frac{1}{2}\mathbf{H}^0 f(\mathbf{x}), \qquad \mathbf{x}\in M, \tag{20}$$

and, by analogy with Sec. 1, expressing the magnetic dipole moment through $\mathbf{h}(\mathbf{x})$, we obtain the second variational functional

$$m^*_{\alpha\beta} + m_{\alpha\beta}\{\mathbf{h}_\alpha,\mathbf{h}_\beta\} = \frac{1}{4\pi}\frac{\int_M ds\, f(\mathbf{x})\,\mathbf{a}_\alpha\cdot\mathbf{h}_\beta(\mathbf{x})\int_M ds\, f(\mathbf{x})\,\mathbf{a}_\beta\cdot\mathbf{h}_\alpha(\mathbf{x})}{\int_M ds\int_M ds'\,\mathbf{h}_\alpha(\mathbf{x})\cdot\overleftrightarrow{I}\, g(\mathbf{x},\mathbf{x}')\cdot\mathbf{h}_\beta(\mathbf{x}')}, \tag{21}$$

where the quantity $m^*_{\alpha\beta} = p^*\delta_{\alpha\beta}$ represents that part of the elements of the magnetic polarizability tensor which is associated with the Kirchhoff approximation. Expression (21) is stationary on the class of functions ensuring convergence of the integrals. The true distributions $\mathbf{h}^0_\alpha$ satisfy the integral equations

$$\int_M ds'\, g(\mathbf{x},\mathbf{x}')\,\mathbf{h}_\alpha(\mathbf{x}') = -\frac{1}{2} f(\mathbf{x}) H^0_\alpha \mathbf{a}_\alpha, \qquad \mathbf{x}\in M \tag{22}$$

and the conditions at the edge $\mathbf{n}(\mathbf{x})\cdot\mathbf{h}(\mathbf{x}) \sim \xi^{-1/2}$, $\mathbf{t}(\mathbf{x})\cdot\mathbf{h}(\mathbf{x}) \sim \text{const} + \xi^{1/2}$ [5]. The diagonal elements $m_{\alpha\alpha}\{\mathbf{h}_\alpha\}$, when the exact solutions are substituted, have maxima or minima, so that, generally speaking, nothing can be said about the off-diagonal ones.

Thus, in an arbitrary basis there always exist two-sided variational estimates for the diagonal elements. If the aperture has at least one axis of symmetry, then one of the coordinate axes should be directed along it; then $\mathbf{a}_\alpha$ will serve as the principal axes of the tensor $\overleftrightarrow{\mathbf{m}}$ ($m_{12} = 0$). In the case of an arbitrary aperture, the available two-sided estimates for the diagonal components of the tensor $\overleftrightarrow{\mathbf{m}}$ in two bases rotated relative to one another by an angle $\pi/4$ make it possible to indicate upper and lower bounds also for the off-diagonal components in either of these bases. Indeed, upon rotation of the coordinate axes by an angle $\pi/4$, the trial functions in the new basis are expressed through the sum and difference of the trial functions in the old basis, while the components of the symmetric tensor $\overleftrightarrow{\mathbf{m}}$ transform according to the law

$$\begin{Vmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{Vmatrix} \to \begin{Vmatrix} \dfrac{m_{11}+m_{22}}{2} + m_{12} & \dfrac{m_{22}-m_{11}}{2} \\ \dfrac{m_{22}-m_{11}}{2} & \dfrac{m_{11}+m_{22}}{2} - m_{12} \end{Vmatrix}, \tag{23}$$

from which the validity of the assertion follows.

From physical considerations it is clear that the results obtained retain their meaning also in the case when the aperture is cut in the common wall of two arbitrary volumes; in this case it is necessary that, in the vicinity of the aperture, $x \sim l$, the principal radii of curvature be much greater than l.

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Note: Figure translations are in progress. See original paper for figures.