



---

Soviet-era science, translated into English

# TOPOLOGICAL EMBEDDINGS OF MANIFOLDS

1969

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.97755>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 513.836

**MATHEMATICS**

**A. V. CHERNAVSKII**

## **TOPOLOGICAL EMBEDDINGS OF MANIFOLDS**

*(Presented by Academician P. S. Aleksandrov on 30 XII 1968)*

1. The purpose of this note is twofold. First, in Sec. 2 we apply here the method of the paper <sup>(1)</sup> to prove the local  $p$ -connectedness of the space of locally flat embeddings (of codimension different from two). On the other hand, in Sec. 4 new proofs are given of two recently obtained results: the author's result on the  $(n - 3)$ -stability of homeomorphisms of  $R^n$  and the Bryant-Seebeck result on the local flatness of locally simply connected embeddings of codimension greater than two <sup>(2, 3)</sup>. The proofs of both these results relied on Homma's theorem <sup>(4)</sup> on piecewise-linear approximation of embeddings (the main part of Homma's argument was published by him in <sup>(5)</sup>). However, it turned out that Homma's proof is incomplete,\* and the author does not know whether it is possible to complete it by Homma's method. But for some simple manifolds, in particular cells and spheres of codimension greater than two, piecewise-linear approximation of embeddings can be obtained in another way, close to Connell's method from <sup>(7)</sup>. We give in Sec. 3 an outline of this proof. Although, for carrying out the proofs given in <sup>(2, 3)</sup>, it is sufficient to be able to approximate embeddings of cells, in Sec. 4 we give new proofs of these results based on the approximation theorem and on the main result of this note—Theorem 1; in this form the arguments become simpler and more natural. Finally, Theorem 4 also yields the theorem on the union of cells (see <sup>(8, 9)</sup>), although only in codimensions greater than two.

In this note we restrict ourselves to brief sketches of the proofs.

We regard manifolds as metrizable and equipped with fixed metrics  $\rho$ . By  $\text{Int } M$  and  $\partial M$  are denoted the interior and the boundary of the manifold  $M$ . By  $I^m(r)$  is denoted the standard cube  $\{x \in R^m \mid |x_i| \leq r, 1 \leq i \leq m\}$ ,  $I^m(1) = I^m$ , and  $o$  is the center of the cube.

An embedding  $q : M^m \rightarrow N^n$  is a homeomorphic mapping onto a locally closed set in  $N^n$ ;  $n - m$  is called the codimension of the embedding. An embedding is locally simply connected if for every point  $x \in M$  and for every neighborhood  $O$  of the point  $qx$  there is a neighborhood  $O'$  of the point  $qx$  such that every map of

a circle into  $O' \setminus qM$  is null-homotopic in  $O \setminus qM$ . In particular,  $m \neq n - 2$ . An embedding is locally flat if, for every point  $x \in M^m$ , it extends to an embedding of a neighborhood of  $x$ , taken in  $M_1 \times I^{n-m}$ , where  $M \subset M_1$ ,  $[M_1 \setminus M] \approx \partial M \times [0, 1]$ , and  $M_1 = M_1 \times o$ . An embedding is proper if  $q(\partial M) \subset \partial N$  and  $q(\text{Int } M) \subset \text{Int } N$ .

By a compact set of embeddings is meant a continuous mapping  $Q : Z \times M \rightarrow Z \times N$ , where  $Z$  is compact and for each point  $z \in Z$  the mapping  $Q_z = Q|_{z \times M}$  is an embedding into  $z \times N$ ; if  $Z$  is a cell or a sphere, one speaks of a cell or a sphere of embeddings. An isotopy  $M$  is a segment of homeomorphisms of  $M$ .

By  $\text{Fr } X$  and  $[X]$  are denoted the boundary and closure; by  $O_\varepsilon(X)$ , the  $\varepsilon$ -neighborhood of  $X$ ;  $\Lambda$  denotes the empty set.

## 2. The space of locally flat embeddings.

**Theorem 1.** *Let  $q : M^m \rightarrow N^n$  be a locally simply connected embedding,  $n \neq m - 2$ ,  $n \geq 5$ , and let  $\varepsilon(x)$  be a nonnegative function on  $N$ , strictly positive in some neighborhood of  $qM$  in  $N$ .*

*Then there exists a strictly positive function  $\delta(x)$  on  $M$  such that*

---

\* The gap in <sup>(5)</sup> was pointed out to the author by M. Shtan' ko; see also <sup>(6)</sup>.

for any compact set of locally flat regular embeddings  $Q : Z \times M \rightarrow Z \times N$ , where  $\rho(q(x), Q_z(x)) < \delta(x)$  for all  $x \in M$  and all  $z \in Z$ , and for  $z_0 \in Z$  there is a compact set of isotopies

$$\Phi : Z \times N \times [0, 1] \rightarrow Z \times N \times [0, 1],$$

where

$$\Phi(z, y, t) = (z, (\Phi_z)_t y, t),$$

such that  $\Phi_{z_0}$  is the identity isotopy,  $(\Phi_z)_1 Q_z = Q_{z_0}$  for all  $z \in Z$ , and

$$\rho(x, (\Phi_z)_t x) < \varepsilon(x)$$

for all  $z \in Z$ ,  $x \in N$ , and  $t \in [0, 1]$ .

**Remark 1.** Taking in Theorem 1, as  $Z$ , an arbitrary sphere, we obtain that the space of locally flat embeddings  $M$  in  $N$  is locally  $p$ -connected for all  $p = 0, 1, 2, \dots$  in the majorant topology (cf. <sup>(1)</sup>). In what follows we restrict ourselves to consideration of the case  $p = 0$ .

**Remark 2.** The theorem can be formulated so that it remains valid also for irregular embeddings. We shall not do this here, but, on the contrary, henceforth assume that  $M$  and  $N$  have no boundary.

**Remark 3.** The theorem can be substantially strengthened. Namely, if it is assumed that all embeddings  $Q_z$  coincide on some closed set  $X \subset M$ , then one

may require that all isotopies  $\Phi_z$  be the identity on the image of any closed subset  $X'$  lying strictly inside  $X$  (cf. Proposition (A) in <sup>(1)</sup>).

**Remark 4.** The dimensional restrictions in Theorem 1 arise because, using first one result of Bryant and Seebeck <sup>(10,11)</sup>, we achieve that certain additional requirements are fulfilled for the embeddings, and only after this do we apply the technique of <sup>(1)</sup>, which by itself imposes no restrictions on dimension. In particular, if it is known that these conditions are fulfilled from the very beginning, then the result will also be true in codimension two.

We shall now give this result of Bryant–Seebeck <sup>(10)</sup> in the form in which we shall need it both in the proof of Theorem 1 and in our proof of Theorem 4 –another result of the same authors. The proof, essentially in this form, was given in <sup>(11)</sup>, where the result was extended to higher codimensions.

**Bryant–Seebeck Theorem.** Let

$$q : M^m \rightarrow R^n, \quad n - m \neq 2, \quad n \geq 5,$$

be a locally 1-connected embedding of a compact polyhedron. For every  $\varepsilon > 0$  there exist numbers  $\eta > 0$  and  $\delta > 0$  such that, if  $q'$  is a  $\delta$ -close to  $q$  locally 1-connected embedding and  $V$  is a neighborhood of  $M' = q'M$ , then there exists a neighborhood  $W(M')$  and an  $\varepsilon$ -isotopy

$$\varphi : R^n \times [0, 1] \rightarrow R^n \times [0, 1]$$

such that: 1)  $(\varphi)_0 = e$ ; 2)  $\varphi$  is the identity on  $R^n \setminus O_\varepsilon(M)$  and on  $W(M')$ ; 3)  $(\varphi)_1 V \supset O_\eta(qM)$ .

**Sketch of the proof of Theorem 1.** The proof proceeds in three steps. The first step is a reduction “to the local case,” formulated below as a lemma. The reduction and the lemma are more or less direct analogues of the corresponding places in <sup>(1)</sup>.

**Lemma.** Let

$$q : I^m(2) \rightarrow R^n$$

be a locally 1-connected embedding,

$$m \neq n - 2, \quad n \geq 5.$$

For every  $\varepsilon > 0$  and for every  $k$ ,  $1 \leq k \leq m$ , there exists a  $\delta > 0$  such that, if  $q'$  and  $q''$  are two locally flat embeddings of  $I^m(2)$  in  $R^n$ ,  $\delta$ -close to  $q$  and coinciding on

$$I^m(2) \setminus I^k \times I^{m-k}(2),$$

then one can construct an  $\varepsilon$ -isotopy

$$\varphi : R^n \times [0, 1] \rightarrow R^n \times [0, 1],$$

which is the identity outside  $O_\varepsilon(qI^m)$  and on

$$q(I^m(2) \setminus (I^k \times I^{m-k}(2))),$$

such that  $(\varphi)_0 = e$  and  $(\varphi)_1 q'' = q'$  on  $I^m$ .

The second step consists in proving, under the hypotheses of the lemma, the following assertion:

For every natural number  $r$  and for every real number  $\xi > 0$  there exists a  $\delta > 0$  such that, if  $q'$  and  $q''$  are two  $\delta$ -close to  $q$  locally flat embeddings of  $I^m(2)$  in  $R^n$ , then we can extend each of them to an embedding of  $I^n(2)$  in  $R^n$  so that the following conditions are satisfied: 1)  $q'O_i \subset q''O_i \subset q'O_{i+1}$ , where  $O_i$  is the  $1/i$ -neighborhood of  $I^m(1, 5)$  in  $I^n(2)$ ; 2) the diameter of the image of each perpendicular dropped from  $\partial I^n(2)$  to  $I^m(2)$ , under each extension, is less than  $\xi$ .

For the proof we first extend the embeddings  $q'$  and  $q''$  to  $I^n(2)$  so that the lengths of the images of the perpendiculars indicated in 2) are very small, and then apply the Bryant–Seebeck theorem alternately  $r$  times to the embedding  $q'$  and  $r$  times to the embedding  $q''$ .

If both requirements 1) and 2) are satisfied for the extensions  $q_1$  and  $q_2$  of the given embeddings, then we consider the embedding  $\tilde{q} = q_2^{-1}q_1$  and construct for it an isotopy, identical on  $R^n \setminus \tilde{q}I^n(2)$  and on  $I^m(2) \setminus I^k \times I^{m-k}(2)$ , and taking  $\tilde{q}$  into an embedding identical on  $I^m$ . The lemma follows easily from this. To construct such an isotopy we apply the technique of the article <sup>(1)</sup>, with the corresponding changes. This is the third step of the proof.

### 3. Approximation of cells and spheres

**Theorem 2.** *Every embedding  $q : M^m \rightarrow R^n$ , where  $m \leq n-3$  and  $M$  is a cube  $I^m$  or a sphere  $\partial I^{m+1}$ , can be approximated arbitrarily closely by a piecewise linear embedding.*

**Sketch of proof.** For a given natural number  $r$ , take concentric cubes

$$I_0^m = I^m, \quad I_1^m, \dots, I_r^m,$$

where

$$I_i^m = I^m \left( \frac{r-i+1}{r} \right).$$

Using the inductive assumption and Zeeman's technique <sup>(12)</sup>, we construct polyhedra  $D_i$  in  $R^n$  and piecewise linear embeddings  $q_i : \partial I_i^m \rightarrow R^n$  such that: 1)  $D_i \cap D_{i+1} = S_{i+1}$ , where  $S_i = q_i \partial I_i^m$ ,  $0 \leq i \leq r-1$ ; 2)  $D_i$  combinatorially collapses to  $S_i$ ,  $0 \leq i \leq r-1$ , and  $D_r$  to  $q_0$ ; 3)  $D_i \subset O_\delta(q_i I_i^m \setminus I_{i+1}^m)$ ; 4) for each point  $x \in S_{i+1}$ , the path of the point  $x$  under the deformation of  $D_i$  into  $S_i$ , determined by the combinatorial collapse of  $D_i$  onto  $S_i$ , lies in

$$O_\delta(q(R_x \cap (I_i^m \setminus I_{i+1}^m))),$$

where  $\delta > 0$  is the given number, and  $R_x$  is the radius of  $I^m$  drawn through  $q_{i+1}^{-1}x$ . Taking a regular neighborhood  $ND_r$  relative to  $D_{r-1}$  (see <sup>(14)</sup>), we obtain a cell

on whose boundary lies  $S_r$ . According to Zeeman <sup>(15)</sup>, there exists a piecewise linear embedding  $\tilde{q} : I^m \rightarrow R^n$  such that  $q_r = \tilde{q}\pi$ , where  $\pi$  is a homothety of  $I_r^m$  onto  $I^m$ . Each collapse of  $D_i$  onto  $S_i$ ,  $0 \leq i \leq r-1$ , extends to a piecewise linear map

$$\chi_i : R^n \rightarrow R^n,$$

identical outside a small regular neighborhood of  $D_i$  relative to  $D_{i-1}$ , which maps  $R^n \setminus D_i$  homeomorphically onto  $R^n \setminus S_i$ . We replace  $\tilde{q}$  by

$$\tilde{q}' = \chi_0 \chi_1 \cdots \chi_{r-1} \tilde{q}.$$

Obviously,  $\tilde{q}'$  is an embedding on  $\text{Int } I^m$ ,

$$\tilde{q}'(\text{Int } I^m) \cap S_0 = \Lambda, \quad \tilde{q}'(\partial I^m) = S_0.$$

Moreover, if for fixed  $r$  one takes  $\delta$  sufficiently small, then we obtain that the image of each radius  $R$  of the cube  $I^m$  lies in a small neighborhood of the arc  $qR$ . Finally, one may suppose that there is a sequence of concentric cubes  $\bar{I}_i^m$  such that  $\chi_i$  is identical on the image of  $\bar{I}_{i+1}^m$  under the map  $\chi_{i+1} \cdots \chi_{r-1} q$ , and also that

$$\bar{I}_0^m \subset \text{Int } I^m, \quad \bar{I}_{i+1}^m \subset \bar{I}_i^m.$$

Let  $\tau$  be a homeomorphism of  $I^m$  onto itself, taking each radius into itself and such that

$$\tau \bar{I}_i^m = \bar{I}_i^m.$$

Then

$$\hat{q} = \tilde{q}' \tau$$

will be a piecewise linear embedding; moreover, if  $r$  is sufficiently large and  $\delta$  sufficiently small, then  $\hat{q}$  will be an  $\varepsilon$ -approximation of the embedding  $q$  for the given  $\varepsilon > 0$ .

As for the approximation of an embedding of spheres, it is easily obtained from the possibility of approximating cells, and we shall not dwell on it here. With the help of Theorem 3 (§4) and Connell's result <sup>(7)</sup>, one can also obtain an approximation of locally flat open cells, but it is not clear to me how to extend this proof to the case of arbitrary open cells.

#### 4. Consequences

We now obtain from Theorems 1 and 2 new proofs of the main results from <sup>(2,3)</sup>.

**Theorem 3.** *Homeomorphisms  $R^n$  are  $(n-3)$ -stable (see <sup>(2)</sup>).*

If  $h$  is a given homeomorphism, then, using Theorem 2, we take a piecewise linear embedding  $\tilde{h}$ , sufficiently close to  $h|_{\partial I^{n-2}}$ , and extend it to a piecewise linear homeomorphism

$$\hat{h} : R^n \rightarrow R^n$$

according to the theo-

Zeeman's theorem<sup>15</sup>. According to Theorem 1, there exists a stable homeomorphism  $\hat{h}$  such that  $h|_{\partial I^{n-2}} = \hat{h}\hat{q}$ . Then

$$h = (\hat{h}h^{-1}\hat{h}^{-1})(\hat{h}\hat{h}),$$

where  $\hat{h}\hat{h}$  is stable, and  $\hat{h}h^{-1}\hat{h}^{-1}$  is identical on  $\partial I^{n-2}$ .

**Theorem 4.** *Locally one-to-one embeddings  $q : M^m \rightarrow N^n$  for  $m - n \geq 3$  and  $n \geq 5$  are locally flat (see (3)).*

In view of the local character of both these properties, it suffices to carry out the proof for an embedding of  $I^m$  in  $R^n$ .

According to Theorem 2, for every sequence of numbers  $\varepsilon_i \rightarrow 0$  we can choose a sequence of piecewise linear embeddings  $q_i$ , where  $\rho(q_i(x), q(x)) < \varepsilon_i$  for all  $x$ . Piecewise linear embeddings are locally flat<sup>15</sup>, and we may assume that  $q_i$  is extended to a neighborhood  $U$  of the cube  $I^m$  in  $R^n$ . Choose a strictly decreasing sequence of neighborhoods  $U_i$  of the cube  $I^m$  so that

$$I^m = \bigcap_{i=1}^{\infty} U_i \quad \text{and} \quad U \supset U_i \supset [U_{i+1}].$$

The embeddings  $q_i$ , according to Theorem 1, can be chosen so that the embedding  $q_i$  is  $\varepsilon_i$ -isotopic to the embedding  $q_{i+1}$  under an isotopy  $\varphi_i$  which is identical outside the  $\varepsilon$ -neighborhood of  $qI^m$ . According to the theorem of Bryant-Seebeck, we can choose the embeddings  $q_i$  and the numbers  $\varepsilon_i$  successively so that for any neighborhood  $V(q_{iI}^m)$  there is such an  $\varepsilon_i$ -isotopy  $\psi_i$ , fixed outside the  $\varepsilon_i$ -neighborhood of  $qI^m$  and on  $q_{iI}^m$ , that

$$(\psi_i)|_{q_{iU}i} \supset O_{\varepsilon+1}(qI^m).$$

If we now also assume that  $\varepsilon_i < 1/2^i$ , then  $q$  is obtained from  $q_1$  by means of an isotopy  $\Phi$  equal to the limit of the isotopies

$$\Phi_k = \varphi_k \psi_k \varphi_{k-1} \psi_{k-1} \cdots \varphi_1 \psi_1.$$

Namely, for each  $t$ ,  $(\Phi_k)_t$  is the superposition of the homeomorphisms

$$(\varphi_k)_t (\psi_k)_t \cdots (\varphi_1)_t (\psi_1)_t,$$

and

$$(\Phi)_t = \lim_{k \rightarrow \infty} (\Phi_k)_t.$$

Since  $q_1$  is locally flat, the theorem is proved.

Let us now note that an embedding of an  $(n - 3)$ -dimensional polyhedron is locally one-to-one if and only if the embedding of each of its open simplexes is locally one-to-one. This almost obvious remark, together with Theorem 4, leads to the following formulation of the result of<sup>8,9</sup> for  $m \leq n - 3$ :

**Cell-combining theorem.** *If an embedding  $I^m$  in  $R^n$ ,  $n - m \geq 3$ , is locally flat at the points  $I^m \setminus I^{m-1}$  and locally one-to-one on  $I^{m-1}$ , then it is locally flat.*

In conclusion we note the following result, which follows in an obvious way from Theorem 1:

**Covering-isotopy theorem.** *If  $Q : I^p \times M \rightarrow I^p \times N$  is a  $p$ -cell of locally flat embeddings,  $n - m \neq 2$ , and  $o$  is the center of  $I^p$ , then there exists such a cell  $H : I^p \times N \rightarrow I^p \times N$  of homeomorphisms of  $N$  that*

$$H(o \times N) = 1, \quad Q|_{t \times N} = H|_{t \times N} Q|_{o \times M}.$$

Here we assume that  $M$  is compact and that the embeddings are proper, although, of course, with the corresponding changes this result can also be extended to more general classes of embeddings. As for codimension 2, here the result, as is known, is not true, and in order to preserve it one must consider embeddings with normal microbundles or with neighborhoods (cf. <sup>1</sup>).

Steklov Mathematical Institute  
Academy of Sciences of the USSR  
Moscow

Received  
20 XII 1968

## References

- <sup>1</sup> A. V. Chernavskii, DAN, 182, No. 3, 510 (1968).
- <sup>2</sup> A. V. Chernavskii, Matem. sborn., 70, No. 4, 605 (1966).
- <sup>3</sup> J. L. Bryant, C. L. Seebeck, Bull. Am. Math. Soc., 70, No. 2, 378 (1968).
- <sup>4</sup> T. Homma, Lectures, Florida St. Univ. (mimeogr.), 1965.
- <sup>5</sup> T. Homma, Yokohama J. Math., 14, No. 1-2, 47 (1966).
- <sup>6</sup> H. W. Berkowitz, Notices Am. Math. Soc., 15, No. 2, ab 68-T-264, 378 (1968).
- <sup>7</sup> H. W. Berkowitz, *ibid.*, 15, No. 5, ab 658-47, 692 (1968).
- <sup>8</sup> A. V. Chernavskii, Matem. sborn., 75, No. 2, 264 (1968).
- <sup>9</sup> A. V. Chernavskii, DAN, 180, No. 5, 1045 (1968).
- <sup>10</sup> J. L. Bryant, C. L. Seebeck, Quart. J. Math., 19, No. 75, 275 (1968).
- <sup>11</sup> A. V. Chernavskii, DAN, 181, No. 2, 290 (1968).
- <sup>12</sup> E. C. Zeeman, Seminar Notes, Inst. Hautes Ét. (mimeogr.), 1963.
- <sup>13</sup> A. V. Chernavskii, Matem. sborn., 79, No. 2 (1969).
- <sup>14</sup> J. Hudson, E. C. Zeeman, Proc. Lond. Math. Soc., 14, No. 56, 1036 (1964).
- <sup>15</sup> E. C. Zeeman, Ann. Math., 78, No. 3, 501 (1963).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*