V. M. BOROK

1°.** Consider the system of linear differential-difference equations

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Abstract

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MATHEMATICS

V. M. BOROK

ON A BOUNDARY-VALUE PROBLEM IN AN INFINITE LAYER FOR LINEAR DIFFERENTIAL-DIFFERENCE SYSTEMS

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1°. Consider the system of linear differential-difference equations

$$\frac{\partial u_i(x,t)}{\partial t} = \sum_{l=1}^{N} \sum_{(k)(j)} a_{il(k)(j)} \frac{\partial^{(k)} u_l(x+h_j,t)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}, \tag{1}$$

$$i = 1, \dots, N, \qquad x \in R_n, \qquad 0 < t < T,$$

where $\sum_{(k)(j)}$ denotes summation over all possible sets $(k) = (k_1, \dots, k_n)$, where $0 \le k_i \le p$, $\sum k_i \le p$, $p \ge 0$, and over all possible sets $(j) = (j_1, \dots, j_n)$ from a certain finite set, $x = (x_1, \dots, x_n)$, $h_j = (h_{j1}, \dots, h_{jn})$; $a_{il(k)(j)}$ are complex constants. We shall seek a solution of system (1) under the boundary conditions:

$$u_{k_i}(x,0) = 0, \qquad 1 \le k_i \le N;\ i = 1, 2, \dots, r;\ 1 \le r \le N-1,$$

$$u_{m_j}(x,T) = 0, \qquad 1 \le m_j \le N;\ j = 1, 2, \dots, N-r. \tag{2}$$

We shall clarify the question of when our assumptions on the growth of $\bar{u}(x,t) = \{u_1(x,t), \dots, u_N(x,t)\}$ as $\|x\| \to \infty$ guarantee that the solution of problem (1)–(2) is identically zero, and when problem (1)–(2) can have nontrivial solutions.

Let us note that system (1) turns into a system of partial differential equations if $h_j = 0$ for all (j). This case was studied in detail by the author in (^{1}), and we shall not dwell on it here, assuming $\max_j \|h_j\| > 0$.

2°. Denote

$$s = \sigma + i\tau = (\sigma_1 + i\tau_1, \dots, \sigma_n + i\tau_n) = (s_1, \dots, s_n),$$

$$A_{ml}(s) = \sum_{(k)(j)} a_{ml(k)(j)} (is_1)^{k_1} \dots (is_n)^{k_n} \exp\{i(s, h_j)\},$$

$$(s, h_j) = \sum_{k=1}^{n} s_k h_{jk}, \qquad 1 \le m, l \le N,$$

$$P(s) = \| A_{ml}(s) \|_{m,l=1}^{N}, \qquad Q(s,t) = \exp\{tP'(-s)\},$$

where $P'(s)$ is the matrix transpose to the matrix $P(s)$. Deleting in the matrix $Q(s,T)$ the rows with numbers $k_1, \dots, k_r$ and the columns whose numbers differ from $m_1, \dots, m_{N-r}$, we obtain the square matrix $Q(s)$. The **determinant of the boundary-value problem** (1)–(2) is defined as

$$\Delta(s) = \det Q(s).$$

Obviously, the elements of the matrices $P(s)$, $Q(s,t)$, $Q(s)$, and the determinant $\Delta(s)$ are entire functions of

$$s = (s_1, \dots, s_n).$$

In what follows we adopt the notation:

$$Z = \{s : \Delta(s) = 0\}, \qquad \|\operatorname{Im} s\| = \left[\sum_{1}^{n} (\operatorname{Im} s_j)^2 \right]^{1/2}, \qquad a = \inf_{s \in Z} \|\operatorname{Im} s\|.$$

3°. **Theorem 1.** *Let* $a = \infty$ *(i.e.* $\Delta(s) \ne 0$*). Then there exists an* $\alpha > 0$ *such that every solution* $\bar{u}(x,t)$ *of problem (1)–(2) satisfying the condition*

$$|u_j(x,t)| \le C \exp\{\alpha \|x\| \ln \|x\|\}, \qquad j = 1, \dots, N, \tag{3}$$

is identically equal to zero.

We note that this result cannot be substantially improved: problem (1)–(2) with $\Delta(s) \ne 0$ may have a nontrivial solution satisfying, for some $\beta > 0$ ($\beta > \alpha$), the estimate

$$|u_j(x,t)| \le C \exp\{\beta \|x\| \ln \|x\|\}, \qquad j = 1, \dots, N.$$

Theorem 2. *Let* $0 < a < \infty$*. Then every solution of problem (1)–(2) satisfying, for* $\beta < a$*, the condition*

$$|u_j(x,t)| \le C \exp\{\beta \|x\|\}, \qquad j = 1, \dots, N, \tag{4}$$

is identically equal to zero. Problem (1)–(2) has a solution $\bar{u}(x,t) \neq 0$ *satisfying condition (4) with* $\beta > a$*, and if there exists* $s_a \in Z$*,* $\|\operatorname{Im} s_a\| = a$*, then problem (1)–(2) has a solution* $\bar{u}(x,t) \neq 0$ *satisfying (4) with* $\beta = a$*.*

Theorem 3. *Let* $\Delta(\sigma) \neq 0$*, but* $a = 0$*. Then, if* $\bar{u}(x,t)$ *is a solution of problem (1)–(2) and*

$$|u_j(x,t)| \leq C(1 + \|x\|)^M, \qquad j = 1, \dots, N,\ M > 0,\ t \in [0,T],$$

then $\bar{u}(x,t) \equiv 0$*. Whatever* $\varepsilon > 0$ *may be, problem (1)–(2) has a solution* $\bar{u}(x,t) \neq 0$ *satisfying the condition*

$$|u_j(x,t)| \leq C \exp\{\varepsilon\|x\|\}, \qquad j = 1, \dots, N,\ t \in [0,T].$$

We note that if system (1) is a difference system (in x), i.e. $p = 0$, then from the condition $\Delta(\sigma) = 0$ it follows that $a > 0$, and thus the assumptions of Theorem 3 can be realized only for $p > 0$.

Theorem 4. *Let* $\Delta(s) \neq 0$*, but the function* $\Delta(s)$ *has real zeros. Then every solution of problem (1)–(2) belonging to* $L_1(R_n)$ *is identically equal to zero; however, problem (1)–(2) has nontrivial bounded solutions.*

In the case when $\Delta(s) \neq 0$, problem (1)–(2) may even have finite (for each $t \in (0,T)$) nontrivial solutions. On the other hand, examples can be given of systems of the form (1) which have no finite solutions at all except the identically zero one. The following theorem shows in what class of functions problem (1)–(2) with $\Delta(s) \equiv 0$ certainly has a nontrivial solution.

Theorem 5. *Let* $\Delta(s) \equiv 0$*. Then problem (1)–(2) has a solution* $\bar{u}(x,t) \neq 0$ *satisfying, for some* $c > 0$*, the estimate*

$$|u_j(x,t)| \leq C_1 \exp\{-c\|x\| \ln \|x\|\}, \qquad j = 1, \dots, N.$$

4°. The proof of most of the theorems stated above is based, as in (^{1}), on a certain general fact, for the formulation of which we introduce the following notation:

$$\mathfrak{N} = \{j:\ 1 \leq j \leq N,\ j \neq k_l,\ l = 1, \dots, r\};$$

$$\mathfrak{M} = \{j:\ 1 \leq j \leq N,\ j \neq m_l,\ l = 1, \dots, N - r\};$$

$$L\bar{u} \equiv \partial\bar{u}/\partial t - P\bar{u},$$

where P is an $(N \times N)$ matrix whose elements are the linear operators P_{ml} $(1 \leq m, l \leq N)$:

$$P_{ml} u_q(x,t) = \sum_{(k)(j)} a_{ml(k)(j)} \frac{\partial^k u_q(x+h_j,t)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}},$$

$$L^* \bar{u} \equiv -\partial \bar{u}/\partial t - P^* \bar{u},$$

P^* is the matrix formally adjoint to P (i.e.

$$P^* = \|P^*_{ml}\|^N_{m,l=1}, \qquad P^*_{ml} u_q(x,t) = \sum_{(k)(j)} (-1)^{k_1+\dots+k_n} a_{lm(k)(j)} \frac{\partial^k u_q(x-h_j,t)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}).$$

Theorem 6. Let $\Phi = \{\varphi(x);\, x \in R_n\}$ be a linear topological space of functions, dense in some linear normed space E; let E' be the space conjugate to E. Then, if the following conditions are satisfied:

a) for arbitrary functions $\varphi_j(x) \in \Phi$, $j \in \mathfrak{N}$, the problem

$$v_j(x,0) = \varphi_j(x), \quad j \in \mathfrak{N}; \qquad v_j(x,T) = 0, \quad j \in \mathfrak{M},$$

has a solution $\bar{v}(x,t)$, and for any $t \in [0,T]$, $\bar{v}(x,t) \in E$ (i.e. $v_j(x,t) \in E$, $j = 1, \dots, N$);

b) the Cauchy problem $L\bar{u}(x,t) = 0$, $\bar{u}(x,0) = 0$, $\bar{u}(x,t) \in E'$, $0 \leq t \leq T$, has only the solution $\bar{u}(x,t) \equiv 0$,

then every solution of problem (1)–(2) $\bar{u}(x,t) \in E'$, $t \in [0,T]$, is identically equal to zero.

In the proof of Theorem 1, as the space $E = E_\alpha$, one chooses the space of functions satisfying the condition

$$\|f(x)\| = \int_{R_n} |f(x)| \exp\{\alpha \|x\| \ln \|x\|\}\, dx < \infty,$$

and in the role of Φ there appears the countably normed space

$$\Phi_{AB} = \left\{ \varphi(x) : \; |x^k \varphi^{(q)}(x)| \leq C_{\varphi\delta} (A+\delta)^k (B+\rho)^q \left(\frac{k}{\ln k}\right)^k (\ln q)^{2q} \right\},$$

$$k = (k_1, \dots, k_n), \qquad q = (q_1, \dots, q_n), \qquad k_i, q_i = 0, 1, 2, \dots;$$

here A and B are fixed, appropriately chosen numbers, while $\delta > 0,\ \rho > 0$ are arbitrary,

$$(\ln q)^{2q} = (\ln q_1)^{2q_1} \cdots (\ln q_n)^{2q_n}; \qquad \left(\frac{k}{\ln k}\right)^k = \left(\frac{k_1}{\ln k_1}\right)^{k_1} \cdots \left(\frac{k_n}{\ln k_n}\right)^{k_n}.$$

The space Φ_{AB} belongs to the class of spaces of type $S_{a,\eta}^{b,q}$, introduced by I. M. Gel' fand and G. E. Shilov (^{2}, Ch. IV, Appendix I). The question of the nontriviality of such spaces was studied by K. I. Babenko (^{3}).

In the proof of the first part of Theorem 2,

$$E = E^\beta = \left\{ f(x):\ \|f(x)\| = \int_{R_n} |f(x)| \exp\{\beta\|x\|\}\, dx < \infty \right\},$$

$$\Phi = \Phi_\gamma^B = \{\varphi(x):\ |\varphi^{(q)}(x)| \le C_\varphi B^q (\ln q)^q \exp\{-\gamma\|x\|\}\}$$

with suitable $\gamma > 0$ and $B > 0$.

In the proof of the first part of Theorem 3,

$$E = \left\{ f(x):\ \|f(x)\| = \int_{R_n} |f(x)|(1 + \|x\|)^M\, dx < \infty \right\},$$

$\Phi = F(K)$ is the space of Fourier transforms of finite infinitely differentiable functions.

The fact of nontriviality of the spaces Φ_{AB} and Φ_γ^B is established with the aid of the aforementioned theorem of K. I. Babenko (^{3}); the proof of the density of the embeddings $\Phi_{AB} \subset E_\alpha$, $\Phi_\gamma^B \subset E^\beta$ is carried out on the basis of the criterion set forth in (^{2}), pp. 278–279. The fulfillment of condition a) of Theorem 6 is established by the method of Fourier transforms; condition b) is effected by virtue of the results of (^{4}) (see also (^{5}), Ch. II, Appendix I).

The first assertion of Theorem 4 is proved directly without applying Theorem 6. If $\bar{v}(s,t)$ is the Fourier transform of the solution $\bar{u}(x,t)$ of problem (1)–(2), existing under the assumptions of Theorem 4, then one can show that $\bar{v}(s,t) \equiv 0$ for $s \in Z$. Hence it follows that $\bar{u}(x,t) \equiv 0$.

To prove the second part of each of Theorems 2, 3, and 4, the corresponding solution of problem (1)–(2) is constructed in the form

$$\bar{u}(x,t) = \exp\{-i(s,x)\}\bar{Z}(t), \qquad \bar{Z}(t) \ne 0,$$

where $s \in Z$ and $\beta > \|\operatorname{Im} s\| > a$ (or $s = s_a$) in Theorem 2, $0 < \|\operatorname{Im} s\| < \varepsilon$ in Theorem 3, and $\operatorname{Im} s = 0$ in Theorem 4.

The proof of Theorem 5 can be carried out according to the scheme of the proof of Theorem 13 in [1].

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Kharkov State University
named after A. M. Gorky

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.