A CONSTRUCTIVE CHARACTERIZATION OF A CLASS OF FUNCTIONS ON THE SPHERE WITH INTEGRABLE SQUARE

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Abstract

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MATHEMATICS

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A CONSTRUCTIVE CHARACTERIZATION OF A CLASS OF FUNCTIONS ON THE SPHERE WITH INTEGRABLE SQUARE

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Let S be the sphere of unit radius centered at the origin of three-dimensional Euclidean space; let P be a point of the sphere S with spherical coordinates θ, φ. Let $f(P) \in L_2(S)$, and let $E_n^{(2)}(f)$ be the best approximation of the function f on S by spherical sums of order not exceeding $n-1$ in the metric L_2, i.e.

$$E_n^{(2)}(f) = \inf_{S_{n-1}} \|f(P) - S_{n-1}(P)\|_{L_2(S)},$$

where

$$S_n(P) = Y_0(P) + Y_1(P) + \cdots + Y_n(P),$$

and $Y_k(P)$ is the usual notation for a spherical harmonic of order k, so that

$$DY_k(P) = -k(k+1)Y_k(P);$$

here

$$D \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

is the Laplace operator on the sphere.

I. Consider the question of the structural properties of the class of functions satisfying the condition

$$E_n^{(2)}(f) \le \frac{C_1}{n^{2r}}\,\omega\left(\frac{1}{n}\right), \qquad n = 1, 2, \dots, \tag{1}$$

where $C_1 = \text{const}$, and $\omega(\delta)$ is a certain function monotonically tending to zero.

Denote by $L_2^*(S)$ the set of functions from $L_2(S)$ whose Fourier series in spherical functions converge uniformly on S. Let $C(P,h)$ be the circle with center at the point P and spherical radius h on S, and let $f_h(P)$ be the mean value of the function f along the circle $C(P,h)$, i.e.

$$f_h(P) = \frac{1}{2\pi \sin h} \int_{C(P,h)} f(Q)\, dS_Q.$$

Theorem 1. Let $f \in L_2^*(S)$, and let the function $\omega(h) \downarrow 0$ as $h \to 0$ satisfy the (ξ_2)-condition of S. M. Lozinskii ([2]): there exists a constant $C > 1$ such that

$$1 < \varliminf_{h\to 0} \omega(Ch)/\omega(h) \le \varlimsup_{h\to 0} \omega(Ch)/\omega(h) < C^2.$$

Then, in order that inequality (1) hold, it is necessary and sufficient that

$$\|D^r(f - f_h)\|_{L_2(S)} = O(\omega(h)),$$

where D^r denotes the r-th power of the Laplace operator D (r is a nonnegative integer).

An analogous result in the metric of the space $C(S)$ ($C(S)$ is the space of functions continuous on S) was obtained by G. G. Kushnirenko ([3]) for the case $\omega(h) = h^\alpha$ $(0 < \alpha < 2)$, and by the author ([4]) for the function $\omega(h)$ indicated in Theorem 1. The proof of Theorem 1 will follow from Theorems 2 and 3, in which relations are established between the behavior of the Fourier coefficients of the function f and the differential properties of this function.

Let $C_{\nu m}(f)$ be the Fourier coefficients of the function $f(P)$ with respect to the orthonormal system of spherical functions $Y_m^{(\nu)}(P)$ ($\nu = -m, \dots, -1, 0, 1, \dots, m$; $m = 0, 1, 2, \dots$) (see, for example, ([6])). Introduce the notation

$$R_n^{(r)}(f) = \left\{ \sum_{m=n}^{\infty} m^{4r} \rho_m^2(f) \right\}^{1/2}, \qquad \text{where} \qquad \rho_m^2(f) = \sum_{\nu=-m}^{m} C_{\nu m}^2(f),$$

and note that, by virtue of the closedness of the indicated system,

$$R_n^{(0)}(f) = \left\{ \sum_{m=n}^{\infty} \rho_m^2(f) \right\}^{1/2} = E_n^{(2)}(f). \tag{2}$$

Lemma. Let $f \in L_2^*(S)$. Then, for any natural number n, the inequality

$$\sup_{h\leqslant 1/n} \|D^r(f-f_h)\|_{L_2(S)} \leqslant \frac{C_2}{n^4}\sum_{k=1}^{n} k^3\{R_k^{(r)}(f)\}^2,$$

holds, where the constant C_2 depends only on r.

Using this lemma, one proves

Theorem 2. Let $f \in L_2^*(S)$, and let the function $\omega(h)$ satisfy the $(\mathfrak{L}_2)$-condition of S. M. Lozinskii. Then, if

$$\left\{\sum_{m=n}^{\infty} \rho_m^2(f)\right\}^{1/2} = O\left(n^{-2r}\omega\left(\frac{1}{n}\right)\right),$$

then

$$\|D^r(f-f_h)\|_{L_2(S)} = O(\omega(h)), \qquad 0 < h \leqslant 1.$$

It is said that a function $\omega(h) \downarrow 0$ $(h \to 0)$ satisfies the (B)-condition of N. K. Bari ([2]), if

$$\sum_{k=n+1}^{\infty} \frac{1}{k}\omega\left(\frac{1}{k}\right) = O\left(\omega\left(\frac{1}{n}\right)\right).$$

Theorem 3. Let $f \in L_2^*(S)$, β be arbitrary, $0 < p \leqslant 2$, and $r \geqslant 0$ an integer. In order that the relation

$$\left\{\sum_{k=n}^{\infty} k^{\beta}\rho_k^p(f)\right\}^{1/p} = O\left(n^{-\varkappa}\omega\left(\frac{1}{n}\right)\right),$$

where

$$\varkappa = 2r + \frac{1}{2} - \frac{1}{p}(1+\beta) \geqslant 0,$$

hold, it is sufficient that

$$\|D^r(f-f_h)\|_{L_2(S)} = O(\omega(h)),$$

where $\omega(h) \downarrow 0$ $(h \to 0)$, if $\varkappa > 0$, and $\omega(h)$ satisfies the (B)-condition of N. K. Bari, if $\varkappa = 0$ and $p \geqslant 1$.

We note that condition $(\mathfrak{L}_2)$ entails (see [2]) condition (B), and by virtue of this Theorem 1 follows from Theorems 2, 3 and equality (2).

Theorem 3 does not cover the case when $r = \beta = 0$, $p = 1$. In this case the following theorem holds, which is an analogue of G. Lorentz' s theorem [1].

Theorem 4. Let $f \in L_2^*(S)$ and $\omega_f^{(2)}(\delta) = O(\delta^\alpha)$, $\alpha > 1/p - 1/2$ $(0 < p \le 2)$, where

$$\omega_f^{(2)}(\delta) = \sup_{h \le \delta} \|f - f_h\|_{L_2(S)}.$$

Then

$$\left\{ \sum_{k=n}^{\infty} \rho_k^p(f) \right\}^{1/p} = O\left(n^{-\alpha - 1/2 + 1/p}\right). \tag{3}$$

If $\alpha > 1/2$, then Theorem 4 can be applied with $p = 1$, and since in this case the right-hand side of (3) tends to zero as $n \to \infty$, Theorem 4 implies

Corollary 1. Let, for $f \in C(S)$, $\omega_f(\delta) = O(\delta^\alpha)$, $\alpha > 1/2$, where

$$\omega_f(\delta) = \sup_{h \le \delta} \|f - f_h\|_{C(S)}.$$

Then

$$\sum_{n=0}^{\infty} \sum_{\nu=-n}^{n} |C_{\nu n}(f)| < \infty. \tag{4}$$

The latter can also be proved in another way (see Corollary 2). We note that from the convergence of the series (4) there follows the absolute convergence almost everywhere on S of the Laplace series of the function f, i.e. convergence almost everywhere on S of the series

$$\sum_{n=0}^{\infty} |Y_n(P)|, \quad \text{where} \quad Y_n(P) = \frac{2n+1}{4\pi} \iint_S f(Q) P_n(\cos PQ)\, dQ;$$

$P_n(x)$ is the Legendre polynomial. This result generalizes and refines the corresponding result from (5), obtained by another method.

II. Consider the question of conditions for convergence of series of the form

$$\sum_{n=1}^{\infty} n^{\beta} \rho_n^{\alpha}(f). \tag{5}$$

Theorem 5. Let $f \in L_2^*(S)$ and let β be arbitrary, $0 < \alpha < 2$, $N > 1$ an integer. Then the convergence of the series

$$\sum_{n=1}^{\infty} N^{\,n(\beta+1-\alpha/2)} \left\{ \omega_f^{(2)} \left(\frac{1}{N^n} \right) \right\}^{\alpha}$$

implies the convergence of the series (5).

Corollary 2. Let $f \in L_2^*(S)$ and let β be any number, $0 < \alpha < 2$. Then the convergence of the series

$$\sum_{n=1}^{\infty} n^{\beta-\alpha/2} \left\{ \omega_f^{(2)} \left(\frac{1}{n} \right) \right\}^{\alpha}$$

implies the convergence of the series (5).

The result formulated in this corollary (which contains Corollary 1), for $\beta = 0,\ \alpha = 1$, is an analogue of the corresponding result on the absolute convergence of trigonometric Fourier series (see (1), p. 612).

III. Suppose the behavior is known of the Fourier series of a function f with respect to an orthonormal system of spherical functions. Let us consider how the Fourier series of a function g, connected with f by the relation

$$\|g - g_h\|_{L_2(S)} \le \|f - f_h\|_{L_2(S)}. \tag{6}$$

Theorem 6. Let f and g be functions from $L_2^*(S)$, connected with each other by relation (6), and let $\rho_n(f) \le \gamma_n$. Then, if for some α $(0 < \alpha < 2)$

$$\sum_{n=1}^{\infty} n^{-5/2\alpha} \left\{ \sum_{k=1}^{n} k^4 \gamma_k^2 \right\}^{\alpha/2} + \sum_{n=1}^{\infty} n^{-\alpha/2} \left\{ \sum_{k=n+1}^{\infty} \gamma_k^2 \right\}^{\alpha/2} < \infty,$$

then

$$\sum_{n=1}^{\infty} \rho_n^{\alpha}(g) < \infty.$$

On the basis of this theorem one proves

Theorem 7. Let f and g be functions from $L_2^*(S)$, connected with each other by relation (6), and let $\rho_n(f) \le \gamma_n$, where

$$\gamma_n \downarrow 0, \qquad \sum_{n=1}^{\infty} \gamma_n^{\alpha} < \infty, \qquad \frac{2}{5} < \alpha < 2.$$

Then

$$\sum_{n=1}^{\infty} \rho_n^{\alpha}(g) < \infty.$$

In addition to the works mentioned above, we indicate works (7-12), where analogous questions are considered for the one-dimensional case. In these same works one can find references to extensive literature.

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