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MATHEMATICS

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ON A GENERALIZATION OF KAKUTANI' S FIXED POINT THEOREM

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In the work (^{1}), I. L. Glicksberg gave a generalization of Kakutani' s fixed point theorem for a closed mapping of a convex compact set to the case of a locally convex linear topological space. In the present note this theorem is extended to a broader class of mappings.

Let X be a Hausdorff linear topological space and let S be a subset of X. A mapping $\Phi : S \to X$ which carries points into nonempty sets is called closed if its graph

$$\bigcup_{x \in S} (x, \Phi(x))$$

is closed in $X \times X$. In terms of generalized sequences this definition is equivalent to one of the following assertions (^{1}):

a) if $x_\delta \to x_0,\ y_\delta \in \Phi(x_\delta),\ y_\delta \to y_0$, then $y_0 \in \Phi(x_0)$;

b) if $x_\delta \to x_0,\ y_\delta \in \Phi(x_\delta)$, and y_0 is a cluster point of the generalized sequence $\{y_\delta\}$, then $y_0 \in \Phi(x_0)$.

Glicksberg' s result is formulated as follows. If Φ is a closed mapping in a locally convex Hausdorff linear topological space X, carrying points of a compact convex set $S \subset X$ into convex subsets of S, then Φ has a fixed point.

Everywhere below X is a locally convex Hausdorff linear topological space. As a fundamental system of neighborhoods of a point $x \in X$ we shall consider the system

$$\mathcal{V}(x) = \{x + V \mid V \in \mathcal{V}(0)\},$$

where $\mathcal{V}(0)$ is the system of all open, convex, and symmetric neighborhoods of zero. In those cases where $\mathcal{V}(x)$ (respectively $\mathcal{V}(x) \times \mathcal{V}(y)$) is considered as a directed set (^{2}), it is assumed that $U_1 \leq U_2,\ U_i \in \mathcal{V}(x),\ i = 1, 2$ (respectively $(U_1, V_1) \leq (U_2, V_2),\ (U_i, V_i) \in \mathcal{V}(x) \times \mathcal{V}(y),\ i = 1, 2$), if and only if $U_2 \subset U_1$ (respectively $U_2 \subset U_1$ and $V_2 \subset V_1$).

Lemma 1. *Suppose that for every $x \in S$ the set $\Phi(x)$ is closed and, moreover, for an arbitrary neighborhood of zero U there exists a neighborhood $V \subset \mathcal{V}(x)$*

such that

$$\Phi(V \cap S) \subset \Phi(x) + U.$$

Then the mapping Φ is closed.

Proof. Let $x_\delta \to x,\ y_\delta \in \Phi(x_\delta),\ y_\delta \to y$, and $y \notin \Phi(x)$. By the closedness of $\Phi(x)$, there is a neighborhood of zero $U \in \mathcal{V}(0)$ such that

$$(y + U) \cap \Phi(x) = \emptyset$$

or

$$(y + \tfrac{1}{2}U) \cap (\Phi(x) + \tfrac{1}{2}U) = \emptyset.$$

On the other hand, there exists a neighborhood $V \in \mathcal{V}(x)$ such that from $x_\delta \in V \cap S$ it follows that

$$\Phi(x_\delta) \subset \Phi(x) + \tfrac{1}{2}U,$$

whence $y_\delta \notin y + \frac{1}{2}U$. Consequently, the point y cannot be the limit of the generalized sequence $\{y_\delta\}$. The lemma is proved.

In what follows, the Φ under consideration will be assumed to be a mapping from S into S, where S is a convex compact set, and by $\overline{A}$ and $\overline{\mathrm{co}}A$ we shall mean the closure and the convex closure of the set A.

Let

$$\overline{\Phi}(x) = \bigcap_{V \in \mathcal{V}(x)} \overline{\Phi(V)}$$

and

$$F(x) = \overline{\mathrm{co}}\Phi(x),$$

where

$$\Phi(V) = \{y \mid y \in \Phi(x),\ x \in V \cap S\}.$$

Lemma 2. *The mapping F has a fixed point.*

Proof. By Glicksberg' s theorem, it suffices to show that the mapping F is closed. Choose arbitrarily a point $x \in S$ and an (open) neighborhood $U \in \mathcal{V}(0)$. There exists an (open) neighborhood $V_0 \in \mathcal{V}(x)$ such that

$$\Phi(V_0) \subset \Phi(x) + \tfrac{1}{2}U.$$

If this were not so, then the system consisting of the closed sets

$$S \setminus (\overline{\Phi(x) + \tfrac{1}{2}U})$$

and $\overline{\Phi(V)}$, $V \in \mathcal{V}(x)$, would be centered, which is impossible by compactness of S and the absence of a common point for the sets of this system. Let $y \in V_0$, $V \in \mathcal{V}(y)$, and $V \subset V_0$. Then

$$\Phi(y) \subset \Phi(V) \subset \Phi(V_0) \subset \Phi(x) + \tfrac{1}{2}U \subset \overline{\mathrm{co}\,\Phi(x)} + \tfrac{1}{2}U.$$

Since the closure of the convex set

$$\overline{\operatorname{co}\Phi(x)} + \tfrac{1}{2}U$$

is a convex set contained in

$$\overline{\operatorname{co}\Phi(x)} + U,$$

we have

$$F(y) = \overline{\operatorname{co}\Phi(y)} \subset \overline{\operatorname{co}\Phi(x)} + U = F(x) + U.$$

Therefore, by Lemma 1, the mapping F is closed. The lemma is proved.

For an arbitrary point $x \in X$ and a set $B \subset X$, by $L(x, B)$ we shall denote the set of all points $y \in X$ representable in the form

$$y = x + \lambda(z - x), \qquad z \in B, \quad \lambda \geq 0,$$

i.e.,

$$L(x, B) = \bigcup_{\lambda \geq 0} [x + \lambda(B - x)].$$

It is obvious that if $B \subset A$, then $L(x, B) \subset L(x, A)$.

Definition. A mapping $\Phi : S \to S$ will be called **partially closed** if from $x_\delta \to x$, $y_\delta \in \Phi(x_\delta)$, $y_\delta \to y$ it follows that

$$L(x, y) \cap \Phi(x) \neq \emptyset.$$

Since for any cluster point y of the generalized sequence $\{y_\delta\}$ one can select from $\{y_\delta\}$ a generalized subsequence converging to y, in the preceding definition y may be regarded as a cluster point of $\{y_\delta\}$. In this way we obtain an equivalent definition of a partially closed mapping.

It is clear that every closed mapping is also partially closed.

Theorem. Every partially closed mapping Φ, which maps the points of a compact convex set $S \subset X$ into convex subsets of S, has a fixed point.

Proof. Suppose that for some open neighborhood of zero V_0, for all $x \in S$ one has

$$(x + V_0) \cap \Phi(x) = \emptyset.$$

Denote by $A(x)$ the closure of the set $\Phi(x)$. Obviously,

$$(x + V_0) \cap A(x) = \emptyset.$$

Let $z \in V_0$. Choose an open neighborhood of zero $V \in \mathcal{V}(0)$ such that

$$z + V \subset V_0.$$

If $y \in x + \frac{1}{2}V$, then $x \in y + \frac{1}{2}V$, and

$$x + z + \tfrac{1}{2}V \subset y + z + V \subset y + V_0.$$

Consequently,

$$(x + z + \tfrac{1}{2}V) \cap \Phi(x + \tfrac{1}{2}V) = \emptyset$$

and

$$x + z \notin \overline{\Phi(x + \tfrac{1}{2}V)},$$

whence $x + z \notin \overline{\Phi(x)}$, or

$$(x + V_0) \cap \overline{\Phi(x)} = \emptyset. \tag{1}$$

We now show that

$$\overline{\Phi(x)} \subset L(x, \Phi(x)). \tag{2}$$

Indeed, let $y \in \overline{\Phi(x)}$. For an arbitrary

$$\delta = (U, V) \in \mathcal{V}(x) \times \mathcal{V}(x)$$

we have $y \in \Phi(V)$, i.e.,

$$U \cap \Phi(V) \neq \emptyset.$$

Let $x_\delta \in V$, $y_\delta \in U \cap \Phi(x_\delta)$. Then $x_\delta \to x$, $y_\delta \to y$, and, by partial closedness of the mapping Φ,

$$L(x, y) \cap \Phi(x) = \emptyset,$$

i.e., for some $\lambda > 0$ we shall have

$$z = x + \lambda(y - x) \in \Phi(x),$$

or

$$y = x + \frac{1}{\lambda}(z - x) \in L(x, \Phi(x)).$$

Thus, from (1) and (2) we obtain

$$\overline{\Phi}(x) \subset S \cap L(x, \Phi(x)) \setminus (x + V_0). \tag{3}$$

By virtue of the boundedness of the set S, there exists $\lambda_0 > 0$ such that $\lambda_0(S - x) \subset V_0$. Then for $\lambda < \lambda_0$ we have $x + \lambda(A(x) - x) \subset x + V_0$ and, taking (1) into account, $S \cap [x + \frac{1}{\lambda}(A(x) - x)] = \emptyset$. Consequently,

$$S \cap L(x, \Phi(x)) \setminus (x + V_0) \subset \bigcup_{\lambda_0 \leqslant \lambda \leqslant 1/\lambda_0} [x + \lambda(A(x) - x)] = D. \tag{4}$$

Since the set $A(x)$ is convex, D is also convex. Moreover, being the image of the compact set $[\lambda_0, 1/\lambda_0] \times A(x)$ under the continuous mapping $(\lambda, z) \to x + \lambda(z - x)$, D is closed. Taking (3) and (4) into account, we have $\operatorname{co} \overline{\Phi}(x) \subset D$. Since $x \notin D$, it follows that $x \notin \operatorname{co} \overline{\Phi}(x)$ for every $x \in S$, which contradicts Lemma 2.

Suppose now that for every neighborhood of zero $V \in \mathcal{V}(0)$ there exists a point $x_V \in S$ for which

$$(x_V + V) \cap \Phi(x_V) \neq \emptyset.$$

Let $y_V = x_V + z_V \in \Phi(x_V)$, $z_V \in V$. The generalized sequence $\{x_V\}$ has an accumulation point $x \in S$. Consider the set $\Delta = \{(U, V) \mid x_V \in U,\ (U, V) \in \mathcal{V}(x) \times \mathcal{V}(0)\}$. Since x is an accumulation point for $\{x_V\}$, Δ forms a direction cofinal with $\mathcal{V}(x) \times \mathcal{V}(0)$. For $\delta = (U, V) \in \Delta$, putting $x_\delta = x_V$ and $z_\delta = z_V$, we shall have $x_\delta \to x$, $z_\delta \to 0$, and $y_\delta = x_\delta + z_\delta \to x$. Then, by virtue of the partial closedness of the mapping Φ, we have $L(x, x) = x \in \Phi(x)$. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.

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