ON THE ASYMPTOTICS OF THE PREDICTION ERROR IN THE MULTIDIMENSIONAL CASE

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 159.21

MATHEMATICS

V. N. SOLEV

ON THE ASYMPTOTICS OF THE PREDICTION ERROR IN THE MULTIDIMENSIONAL CASE

(Presented by Academician Yu. V. Linnik on 8 VII 1968)

1. Consider a stationary, in the broad sense, sequence $x(n) = (x^1(n), \dots, x^m(n))$. With this sequence there is naturally associated a sequence of subspaces $X(n)$, generated by the random variables $x^1(n), \dots, x^m(n)$. Denote by H_a^b the closed, in mean square, linear span of the subspaces $X(s)$, $a \le s \le b$. The linear space $H_{-\infty}^{\infty}$ is a Hilbert space with scalar product $(x, y) = Ex\bar{y}$.

For two subspaces H_φ, H_ψ of the space $H_{-\infty}^{\infty}$, put

$$\tau(H_\varphi, H_\psi) = \sum_{k,s} |(\varphi_k, \psi_s)|^2,$$

where $\{\varphi_k\}$, $\{\psi_s\}$ are orthonormal bases in the subspaces H_φ, H_ψ, respectively.

Let $\bar{H}(n)$ be the orthogonal complement in the space $H_{-\infty}^{\infty}$ to the subspace H_{-n}^{-1}. Put $\tau_n = \tau(X(0), \bar{H}(n))$. The quantity τ_n can serve as a measure of the accuracy of the linear prediction of vectors $\xi = (\xi^1, \dots, \xi^m)$ with coordinates $\xi^i \in X(0)$ from the past of length n, i.e., from the random variables $x^i(s)$, $i = 1, \dots, m;$ $s = -1, \dots, -n$. In particular, if the random variables $\xi^1, \dots, \xi^m$ form a basis in $X(0)$, then there are constants, independent of n, $0 < m \le M < \infty$ such that $m\tau_n \le \sigma_n^2(\xi) \le M\tau_n$, where $\sigma_n^2(\xi) = \sum_{i=1}^m \sigma_n^2(\xi^i)$, and $\sigma_n^2(\xi^i)$ is the square of the error of the linear prediction of the random variable ξ^i from the past of length n.

If the sequence $x(n)$ is linearly regular (see (1)), then the quantity $\tau_\infty > 0$. It is clear that $\delta_n = \tau_n - \tau_\infty \ge 0$ and $\delta_n \to 0$ as $n \to \infty$. We shall study the rate of decrease of the quantity δ_n to zero depending on the properties of the spectral density (s.d.) $f(\lambda)$ of a linearly regular sequence $x(n) = (x^1(n), \dots, x^m(n))$ of full rank. In the one-dimensional case an analogous problem was considered by I. A. Ibragimov ((2), see also (3,4)).

Theorem 1. In order that, as $n \to \infty$,

$$\delta_n = O(n^{-2p}), \qquad p > 1/2, \quad p \text{ nonintegral}, \tag{1}$$

it is necessary and sufficient that the s.d. $f(\lambda)$ almost everywhere coincide with a continuous matrix-function having a strictly positive determinant, all elements of this matrix having absolutely continuous derivatives $f_{ij}^{(r-1)}(\lambda)$, $r = [p]$, the r-th derivatives $f_{ij}^{(r)}(\lambda) \in L_2(-\pi, \pi)$ and satisfying there a Hölder condition of order $\alpha = p - [p]$; the latter means that

$$\sup_{t \le h} \left(\int_{-\pi}^{\pi} |f_{ij}^{(r)}(\lambda) - f_{ij}^{(r)}(\lambda + t)|^2 \, d\lambda \right)^{1/2} = O(h^\alpha). \tag{2}$$

2. The proof of Theorem 1 is based on the following lemma.

Lemma 1. *If condition (1) is satisfied, the collection $x^i(s)$, $i = 1, \dots, m$; $s = 0, \pm 1, \dots$, forms a Riesz basis in the space $H_{-\infty}$.*

We first derive Theorem 1 from Lemma 1. Let condition (1) be satisfied. Denote by $\Psi^i(s)$, $i = 1, \dots, m$; $s = -1, \dots, -k, \dots$, the system conjugate to the system $x^i(s)$, $i = 1, \dots, m$; $s = -1, \dots, -k, \dots$, in the space $H_{-\infty}^{-1}$. Let $\widetilde{H}(n)$ be the orthogonal complement to the space H_{-n}^{-1} in the space $H_{-\infty}^{-1}$. Obviously,

$$\delta_n = \tau(X(0), \widetilde{H}(n)).$$

Since, by Lemma 1, the collection $x^i(s)$, $i = 1, \dots, m$; $s = -1, \dots, -k, \dots$, forms a Riesz basis in the space $H_{-\infty}^{-1}$, the collection $\Psi^i(s)$, $i = 1, \dots, m$; $s = -1, \dots, -k, \dots$, forms a Riesz basis in the space $H_{-\infty}^{-1}$. Consequently,

$$\delta_n \asymp \sum_{-s=n+1}^{\infty} \sum_{j=1}^{m} \sum_{i=1}^{m} \left| (x^i(0), \Psi^j(s)) \right|^2, \tag{3}$$

where the symbol $a_n \asymp b_n$ means that

$$0 < \varliminf_n \frac{a_n}{b_n} \le \varlimsup_n \frac{a_n}{b_n} < \infty.$$

It also follows from Lemma 1 (see (5)) that there exists an everywhere defined, bounded, invertible, positive Hermitian operator B in $H_{-\infty}^{-1}$ such that

$$x^i(s) = B \Psi^i(s), \qquad i = 1, \dots, m; \; s = -1, -2, \dots \tag{4}$$

From (3) and (4) it follows that

$$\delta_n \asymp \sum_{-s=n+1}^{\infty} \sum_{j=1}^{m} \sum_{i=1}^{m} \left|(x^i(0), x^j(s))\right|^2 .$$

Hence, using the results of approximation theory (see (6), Ch. V), one can show that the matrix $f(\lambda)$ coincides almost everywhere with a continuous matrix whose elements have derivatives of the order indicated in Theorem 1, and that condition (2) is satisfied. In particular, the trace of the mentioned continuous matrix is a bounded function and, consequently, its determinant is strictly positive (see Lemma 1).

The proof of sufficiency is carried out analogously.

3. In the proof of Lemma 1 some results from the theory of orthogonal matrix polynomials are used (see also (7)).

Consider the space $L_2^m(f)$ of matrices $S(\lambda)$ of size $m \times n$ such that

$$\|S\|_f^2 = \operatorname{sp} \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\lambda) f(\lambda) S^*(\lambda)\, d\lambda < \infty.$$

Put

$$(S, P)_f = \frac{1}{2\pi} \int_{-\pi}^{\pi} S f P^*\, d\lambda.$$

The condition

$$(\varphi_n, \varphi_m)_f = \begin{cases} 0, & m \neq n, \\ E, & m = n, \end{cases} \tag{5}$$

uniquely determines a sequence of polynomials orthogonal in the sense of (5), $\varphi_n(z) = k_n z^n + \cdots$, where k_n is a positive Hermitian matrix.

Lemma 2. *All zeros of the function* $\operatorname{Det} \varphi_n(z)$ *lie inside the unit circle.*

Put

$$s_n(x, z) = \sum_0^n \varphi_n^*(x) \varphi_n(z).$$

Lemma 3. Let $g(z)$ be an arbitrary polynomial of degree n. Then

$$\left(g(z), s_n(x, z)\right)_f = g(x), \qquad z = e^{i\lambda}.$$

Lemma 4. The identity

$$s_n(0,z) = \sum_0^n \varphi_n^*(0)\varphi_n(z) = k_n z^n \varphi_n^*(z)$$

holds.

In particular,

$$s_n(0,0) = \sum_0^n \varphi_n^*(0)\varphi_n(0) = k_n^2.$$

Lemma 5. The recurrence formula

$$k_n \varphi_{n+1}(z) = k_{n+1} z \varphi_n(z) + \varphi_{n+1}(0) z^n \varphi_n^*(z)$$

holds.

Lemma 6. Let $g(z)$ be a polynomial of degree n with coefficient E at the leading term. Then

$$\operatorname{Det}(g,g)_f \geqslant \operatorname{Det} k_n^{-2}.$$

The polynomial $k_n^{-1}\varphi_n(z)$ minimizes the quantity $\operatorname{Det}(g,g)_f$ among polynomials of degree n with coefficient E at the leading term.

The following Lemmas 7 and 8 are proved almost in the same way as the corresponding assertions in the one-dimensional case ([2]).

Lemma 7. Under condition (1),

$$\sum |\varphi_k(0)|^2 < \infty.$$

Lemma 8. Under condition (1), there exist constants independent of n, $0 < m \leq M < \infty$, such that

$$mE \leqslant s_n(0,z) s_n^*(0,z) \leqslant ME, \qquad z = e^{i\lambda}.$$

Denote by $G(z)$ the maximal analytic matrix inside the unit disk (see ([1])) such that

$$f(\lambda) = G(e^{i\lambda}) G^*(e^{i\lambda}).$$

It is easy to show, using Lemmas 3 and 7, that under condition (1) the sequence $s_n(0,z)$ converges on the unit circle to the matrix $[G^{-1}(0)]^* G^{-1}(z)$ in the uniform metric. Hence, and from Lemma 8, follows the existence of constants $0 < m \leq$

$M < \infty$ such that the relation $mE \le f(\lambda) \le ME$ is satisfied for almost all $\lambda \in [-\pi, \pi]$. The obtained relation is equivalent to the assertion of Lemma 1.

4. Consider the case when $\delta_n = O(e^{-cn})$, $c > 0$. The following two theorems are proved by the same method as Theorem 1 was proved (cf. Theorems 4 and 5 of ([2])).

Theorem 2. The relation

$$\delta_n = O(e^{-cn}), \qquad c > 0,$$

holds if and only if the s.p. $f(\lambda)$ almost everywhere coincides with a matrix admitting analytic continuation into the strip of values of the complex argument $z = \lambda + i\mu$ of width c and having a strictly positive determinant.

Theorem 3. The relation

$$\delta_n = O(e^{-cn})$$

holds for all $c > 0$ if and only if the analytic continuation of the matrix mentioned in Theorem 2 is an entire function and its determinant is strictly positive.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
18 III 1968

REFERENCES

1. Yu. A. Rozanov, *Stationary Random Processes*, Moscow, 1963.
2. I. A. Ibragimov, *Theory of Probability and Its Applications*, 9, no. 4 (1964).
3. U. Grenander, M. Rosenblatt, *Trans. Am. Math. Soc.*, 76, 112 (1954).
4. G. Baxter, *Math. Scand.*, 10, 2 (1962).
5. N. K. Bari, *Uchen. zap. MGU*, 4, issue 148 (1951).
6. A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Moscow, 1960.
7. V. Gyires, *Magyar tud. acad. mat. kutato int. közl.*, 7, ser. A, 1–2 (1962).