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Abstract

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ON SOME PROPERTIES OF CONCAVE FUNCTIONS OF A SELF-ADJOINT OPERATOR

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1. In recent years there has been increasing interest in the application of methods of functional analysis to the study of various problems of mathematical physics. In this connection, the equations of mathematical physics are written in the form of operator equations, and various properties of solutions (existence, uniqueness, extendability, etc.) are derived from the continuity or complete continuity of the corresponding operators. The operators in various equations are, as a rule, closely connected with one another, and the problem of the interrelation of the continuity or complete-continuity properties of different operators becomes topical.

One type of connection between operators is provided by functional relations of the form $B = \varphi(A)$. Functional relations have been studied in detail in Hilbert spaces; however, going beyond Hilbert space (even for the well-studied spaces $\mathcal{L}_p$) causes considerable difficulties. M. A. Krasnosel' skii (^{1}) was the first to study the properties of fractional powers of a linear operator A^τ $(0 < \tau < 1)$ in the spaces $\mathcal{L}_p$; subsequently the theory of fractional powers was developed rather fully both in the spaces $\mathcal{L}_p$ and in arbitrary Banach spaces (^{2}). More general functional relations in Banach spaces have at present scarcely been studied.

Below we give some results for concave (and closely related) functions $\varphi(A)$, directly adjoining the results for fractional powers.

2. Let A be a linear positive definite self-adjoint operator acting in the space $\mathcal{L}_2(\Omega)$. A function of the operator $\varphi(A)$ can be defined by means of the spectral decomposition

$$\varphi(A) = \int_0^\infty \varphi(\lambda)\, dP_\lambda. \tag{1}$$

For the study of functions of a linear operator outside $\mathcal{L}_2$, Lorentz spaces Λ_α, with norm

$$\|x\|_{\Lambda_\alpha} = \sup \left| \int_\Omega x(s)y(s)\,ds \right|, \tag{2}$$

turn out to be very convenient, where the supremum is taken over all functions $y(s)$ satisfying the inequality

$$\int_D |y(s)|\,ds \le \alpha(\operatorname{mes} D) \tag{3}$$

for every $D \subset \Omega$. It is required here that the functions $\alpha(\lambda)$ and $\lambda/\alpha(\lambda)$ be nondecreasing for $\lambda > 0$ and tend to zero as $\lambda \to 0$. Following P. P. Zabreiko ([3]), we shall call any real function satisfying the listed properties a Φ-function. An important prop-

a property of a Φ-function $\alpha(\lambda)$ is the existence of a concave Φ-function $\bar{\alpha}(\lambda)$, related to $\alpha(\lambda)$ by the inequalities

$$\alpha(\lambda) \le \bar{\alpha}(\lambda) \le 2\alpha(\lambda) \qquad (\lambda > 0). \tag{4}$$

Lorentz spaces have recently been studied rather extensively (see, for example, ([3,4])). In particular, they contain the characteristic functions $\chi(D)$ of all sets $D \subset \Omega$, and $\|\chi(D)\|_{\Lambda_\alpha} = \alpha(\operatorname{mes} D)$. If a linear operator A is bounded on characteristic functions,

$$\|A\chi(D)\|_E \le C\|\chi(D)\|_{\Lambda_\alpha}, \tag{5}$$

then it admits an extension to a continuous operator acting from the whole space Λ_α into E. This last property is the reason for the further use of Lorentz spaces, since many properties of the operator $\varphi(A)$ can be checked directly only on characteristic functions.

3. In the theory of fractional powers of a self-adjoint operator in a Hilbert space H, an important role is played by the so-called moment inequality

$$\|A^\tau x\| \le \|Ax\|^\tau \|x\|^{1-\tau} \qquad (x \in H), \tag{6}$$

first proved by M. A. Krasnosel' skii ([1]).

An analogous inequality can also be proved for more general functions $\varphi(A)$.

Lemma. If $\varphi(\lambda)$ is an arbitrary Φ-function, then for all $x \in H$ the inequality

$$\|\varphi(A)x\|/\|x\| \le 2\varphi(\|Ax\|/\|x\|) \tag{7}$$

holds (for concave Φ-functions the inequality is valid also without the factor 2 on the right-hand side).

It is natural to call inequality (7) the **generalized moment inequality**. It turns into (6) if $\varphi(\lambda) = \lambda^\tau$.

Using the lemma, one can prove the main result on functions of a self-adjoint operator.

Theorem 1. Let the self-adjoint and positive definite operator A in $\mathcal{L}_2$ be at the same time a bounded operator acting from Λ_α into $\mathcal{L}_2$, where $\alpha(\lambda)$ is some Φ-function.

Then an arbitrary Φ-function $\varphi(A)$ of the operator A is a bounded operator acting from Λ_β into $\mathcal{L}_2$, where

$$\beta(\lambda) = \varphi(\alpha(\lambda)/\sqrt{\lambda})\sqrt{\lambda}. \tag{8}$$

If, moreover, A as an operator from Λ_α into $\mathcal{L}_2$ is completely continuous, then $\varphi(A)$ is also completely continuous as an operator from Λ_β into $\mathcal{L}_2$.

It is easy to see that $\beta(\lambda)$ is a Φ-function.

The proof of Theorem 1 does not use any specific properties of the function $\varphi(A)$ except inequality (7). Therefore the result of the theorem will also hold for an arbitrary linear operator B satisfying the inequality

$$\|Bx\|/\|x\| \leq C\varphi(\|Ax\|/\|x\|) \qquad (x \in \mathcal{L}_2). \tag{9}$$

At the same time, the requirement that the operator A be self-adjoint is removed. If, however, one speaks only about the boundedness of the operator B (without complete continuity), it is enough that inequality (9) hold only on characteristic functions. Taking into account that $\|\chi(D)\|_{L_2} = \sqrt{\operatorname{mes} D}$, we arrive at the inequality in the space $\mathcal{L}_2$

$$\|B\chi(D)\| \leq C\varphi(\|A\chi(D)\|/\sqrt{\operatorname{mes} D})\sqrt{\operatorname{mes} D}. \tag{10}$$

If this inequality holds, then from the boundedness of A as an operator from Λ_α into $\mathcal{L}_2$ there follows the boundedness of B as an operator from Λ_β into $\mathcal{L}_2$, where $\beta(\lambda)$ is still defined by equality (8).

There arises the natural question of ways of directly verifying inequality (10). We give one result for integral operators

$$Ax(t) = \int_\Omega K(t,s)x(s)\,ds, \qquad Bx(t) = \int_\Omega L(t,s)x(s)\,ds. \tag{11}$$

Theorem 2. *Suppose that for any finite set of points $s_1, s_2, \dots, s_n \in \Omega$ the inequality*

$$\left\| \frac{L(t,s_1)+\dots+L(t,s_n)}{n} \right\| \leq C_1 \varphi \left(\left\| \frac{K(t,s_1)+\dots+K(t,s_n)}{n} \right\| \right) \tag{12}$$

holds. Then the operators A and B satisfy inequality (10).

5. All the results of the present article will, of course, also be valid if the space Λ_β is replaced by any other Banach space E embedded in Λ_β. The conditions for the embedding into Lorentz spaces were studied in detail by P. P. Zabreiko ([5]); they allow, for example, passage from Lorentz spaces to Orlicz spaces with arbitrarily small "losses." Passage to other spaces without "losses" would be possible if sufficiently fine interpolation theorems of the Stein–Weiss theorem type ([6]) were available.

Unfortunately, the latter theorem, which covers Lorentz spaces Λ_a with functions $a(\lambda) = \lambda^\tau$ $(0 < \tau < 1)$, has not yet been generalized to the case of arbitrary Φ-functions $a(\lambda)$. Therefore its application is essentially limited to fractional powers of an operator, for which one can prove a statement strengthening the known results.

Theorem 3. *Let a linear self-adjoint positive definite operator A act continuously from the space Λ_α, where $\alpha(\lambda) = \lambda^a$, into the space $\mathcal{L}_2$.*

Then the operator A^τ $(0 < \tau < 1)$ acts continuously from the space $\mathcal{L}_\beta$, where $\beta = \lambda^b$, $b = \frac{1}{2} + \tau(a - \frac{1}{2})$, into the space $\mathcal{L}_2$.

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