ON ISOMETRIC OPERATORS IN SPACES OF SUMMABLE ANALYTIC AND HARMONIC FUNCTIONS

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Abstract

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A. I. PLOTKIN

ON ISOMETRIC OPERATORS IN SPACES OF SUMMABLE ANALYTIC AND HARMONIC FUNCTIONS

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Let D be a bounded domain in the complex plane and let σ be plane Lebesgue measure in D. By $L_a^p(D)$ ($L_h^p(D)$) we denote the space of all analytic (respectively, complex-valued harmonic) functions f for which the norm

$$\|f\| = \left(\int_D |f|^p \, d\sigma \right)^{1/p}$$

is finite ($p \geq 1$), and by $B(D)$ the space of all analytic functions bounded in D, with the sup-norm.

The following theorem of Kakutani—Chevalley is known (^{1}):
Let D_1 and D_2 be domains having no AB-removable boundary points (a boundary point is called AB-removable if every function in $B(D)$ admits analytic continuation to some neighborhood of this point). If the algebras $B(D_1)$ and $B(D_2)$ are isomorphic, then the domains D_1 and D_2 are conformally equivalent.

The same conclusion follows from the isometry of the B-spaces $B(D_1)$ and $B(D_2)$ (^{2}).

In the present note we prove analogues of the second of these two theorems for the spaces $L_a^p(D)$ and $L_h^p(D)$, and also for the spaces $E^p(D)$ of V. I. Smirnov (^{3}, ^{4}) (for the definition of $E^p(D)$ for a multiply connected domain see, for example, (^{5})). It turns out that under natural conditions on the domains the isometry of the spaces $L_a^p(D_1)$ and $L_a^p(D_2)$ (or of the spaces $E^p(D_1)$ and $E^p(D_2)$), $p \neq 2$, entails the conformal equivalence of the domains D_1 and D_2. The isometry of the spaces $L_h^p(D_1)$ and $L_h^p(D_2)$, however, entails (under somewhat more restrictive conditions on the domains) the congruence of D_1 and D_2.

The basis for the proof of these results is a fairly general Theorem 1, which apparently is also of independent interest. The method used to prove this theorem is a development of the method first applied by Forelli (^{6}) to describe the general form of isometric operators in the Hardy spaces H^p.

We note that all the theorems formulated below, with obvious changes in the formulations, are also valid for $0 < p < 1$.

1. Let (X_1, σ_1) and (X_2, σ_2) be two spaces with positive normalized measures, and let B_0 be a subalgebra with identity in $L^\infty(\sigma_1)$. Let $1 \leq p < \infty$, $p \neq 2$, and let T be a linear operator mapping B_0 into $L^p(\sigma_2)$.

Lemma 1. Suppose that $T1 = 1$, and assume that to each $f \in B_0$ there correspond two numbers a and δ, $\delta > 0$, such that for all z, $|z| < \delta$,

$$\int_{X_1} |1 + zf|^p \, d\sigma_1 = \int_{X_2} |1 + zTf|^p \, d\sigma_2 + a|z|^p.$$

Then: 1) $a = 0$ for all $f \in B_0$; 2) T maps B_0 into $L^\infty(\sigma_2)$; 3) T is multiplicative on B_0; 4) T is isometric in the metrics of the spaces L^{2k}, $k = 1, 2, ...$, and L^∞.

Lemma 2. Let T be an isometric operator in the L^p-metrics, i.e.

$$\int_{X_1} |f|^p \, d\sigma_1 = \int_{X_2} |Tf|^p \, d\sigma_2$$

for all $f \in B_0$. Let $F = T1$, and let E be such a (measurable) set in X_2 that $F = 0$ σ_2-almost everywhere outside E and $F \neq 0$ σ_2-almost everywhere on E. Then for every $f \in B_0$, $Tf = 0$ σ_2-almost everywhere outside E.

With the help of Lemmas 1 and 2 one proves the following.

Theorem 1. Let T be an isometric operator in the L^p-metrics $(p \neq 2)$. Then T has the form

$$Tf = F\varphi(f),$$

where $F \in L^p(\sigma_2)$ $(F = T1)$ and φ is a homomorphism of the algebra B_0 into $L^\infty(\sigma_2)$, isometric in the L^∞-metrics.

Proof. Let $F = T1$, $d\sigma_3 = |F|^p d\sigma_2$, and let E be such a set as in Lemma 2. Put $T_1 f = Tf/F$ on E and $T_1 f = 0$ outside E. Then $T_1 1 = 1$ (σ_3-almost everywhere), the measure σ_3 is normalized, and, as follows from Lemma 2, T_1 maps B_0 isometrically with the $L^p(\sigma_1)$-metric into $L^p(\sigma_3)$. By Lemma 1, T_1 is a homomorphism of the algebra B_0 into $L^\infty(\sigma_3)$, isometric in the $L^\infty(\sigma_1)$ and $L^\infty(\sigma_3)$ metrics. But, again with the help of Lemma 2, it is easy to show that in this assertion one may replace σ_3 by σ_2, as was required to prove.

2. **Theorem 2.** Let D_1 and D_2 be two bounded domains in the plane and $p \neq 2$. If the spaces $L^p_a(D_1)$ and $L^p_a(D_2)$ are isometrically isomorphic, then the B-algebras $B(D_1)$ and $B(D_2)$ are algebraically and isometrically isomorphic.

Theorem 3. Let the bounded domains D_1 and D_2 have no AB-removable boundary points. Then, if the spaces $L_a^p(D_1)$ and $L_a^p(D_2)$ $(p \neq 2)$ are isometrically isomorphic, the domains D_1 and D_2 are conformally equivalent.

The **proof** follows from Theorem 2 and the Kakutani–Chevalley theorem formulated above.

Using the more complete formulation of the Kakutani–Chevalley theorem (see $(^1)$) and narrowing the class of domains under consideration, one can also obtain the general form of isometric mappings of $L_a^p(D_1)$ onto $L_a^p(D_2)$.

Theorem 4. Let D_1 and D_2 be Jordan domains and let T be an isometric isomorphism of $L_a^p(D_1)$ onto $L_a^p(D_2)$. Then T has the form

$$Tf = b(\tau')^{2/p} f(\tau),$$

where b is a number, $|b| = 1$, and τ is a one-to-one conformal mapping of D_2 onto D_1. Conversely, for any such b and τ, the formula written above defines an isometric isomorphism of $L_a^p(D_1)$ onto $L_a^p(D_2)$.

3. Let now D_1 and D_2 be finitely connected domains bounded by rectifiable Jordan curves. For the spaces $E^p(D_1)$ and $E^p(D_2)$, the analogues of the assertions of § 2 are valid. Namely, the following holds.

Theorem 5. Let T be an isometric isomorphism of $E^p(D_1)$ onto $E^p(D_2)$, $p \neq 2$. Then T has the form

$$Tf = b(\tau')^{1/p} f(\tau),$$

where b is a number, $|b| = 1$, and τ is a one-to-one conformal mapping of D_2 onto D_1 such that $(\tau')^{1/p}$ is a single-valued analytic function in D_2. Conversely, for any such b and τ, the written formula defines an isometric isomorphism of $E^p(D_1)$ onto $E^p(D_2)$.

4. We pass to the consideration of the spaces $L_h^p(D)$.

Lemma 3. Let F be a harmonic function in the domain D, $F \neq 0$, and let g be twice continuously differentiable in D, $g \neq$ const. Suppose that the function Fg^n, for whatever natural n, is harmoni-

analytic in D. Then either both functions F and g are analytic, or both functions $\overline{F}$ and $\overline{g}$ are analytic.

In what follows we shall consider only domains D satisfying the following condition:

(α). The linear span $B(D)$ and $\overline{B}(D)$ ($\overline{B}(D)$ is the set of functions complex-conjugate to functions from $B(D)$) is dense in $L_h^p(D)$.

We note that Jordan domains with sufficiently smooth boundary have property (α).

Lemma 4. Let the domains D_1 and D_2 have property (α), and let T be an isometric isomorphism of $L^p_h(D_1)$ onto $L^p_h(D_2)$ $(p \neq 2)$. Then $T1 = \text{const}$.

Theorem 6. If the bounded domains D_1 and D_2 have the same area, have property (α), and have no AB-removable boundary points, then an arbitrary isometric isomorphism T of $L^p_h(D_1)$ onto $L^p_h(D_2)$ has the form

$$Tf = bf(\tau),$$

where b is a number, $|b| = 1$, and τ is a congruent transformation of D_2 onto D_1. Conversely, for any such b and τ, the formula written above defines an isometric isomorphism of $L^p_h(D_1)$ onto $L^p_h(D_2)$.

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Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.