ON THE REALIZATION OF SPACES OF REGULAR FUNCTIONALS AND SOME OF ITS APPLICATIONS

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 513.88

MATHEMATICS

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ON THE REALIZATION OF SPACES OF REGULAR FUNCTIONALS AND SOME OF ITS APPLICATIONS

(Presented by Academician L. V. Kantorovich on 10 II 1969)

In the theory of vector structures one often uses representations of various spaces in the form of spaces of certain special types, for example, spaces of continuous functions. Such a realization is admitted, in particular, by an arbitrary K-space (see, for example, ([2]), Ch. V). Completely linear functionals on K-spaces admit an integral representation of the same type as linear continuous functionals in the classical L_p $(1 < p < \infty)$; this circumstance makes it possible, in the study of such functionals, to apply widely the apparatus of measure and integration theory. The question of the realization of arbitrary regular functionals (and of spaces of such functionals) is more complicated. The purpose of the present note is to construct one method of realizing spaces of regular functionals and to apply it to Banach structures introduced by Calderon ([3]). Some results are also given on completely linear functionals in a KN-space, based on its unit ball.

We shall use the terminology from the theory of K-spaces (i.e., conditionally complete linear structures) adopted in ([2]). Two elements x, y of a K-space X are called **disjoint** (notation xdy) if $|x| \wedge |y| = 0$. A **unit** 1 of a K-space X is understood in the **weak** sense (in Freudenthal' s sense), i.e., $x \wedge 1 > 0$ for every $x > 0$. A **normal subspace** of a K-space X is any of its linear subspaces X_1 satisfying the condition: if $x \in X_1$, $y \in X$, $|y| \leq |x|$, then $y \in X_1$. If, in addition, in X there are no nonzero elements disjoint from all elements of X_1, then one says that X_1 is a **fundament** in X.

A K-space W is called **extended** if every set of its pairwise disjoint elements is bounded. A bicompact space Q is called **extremal** if the closure of every open set in Q is open-and-closed. For an arbitrary extremal bicompact space Q, the set $C_\infty(Q)$ of all real continuous functions on Q that may take the values $+\infty$ and $-\infty$ only on nowhere dense sets is an extended K-space under the natural partial order and algebraic operations (see ([2]), Ch. V). Every extended K-space W in which a unit 1 is fixed is uniquely realized in the form of a space $C_\infty(Q)$ on a suitable extremal bicompact space Q, if one requires that 1 correspond to the function on Q identically equal to one. Every K-space X is a fundament in some extended K-space W, which is called the **maximal extension** of the space X and which we shall denote by $\mathfrak{M}(X)$.

With every K-space X there are associated two spaces of functionals on X: the space $\widetilde{X}$ of all regular functionals (([2]), p. 267) and the space $\overline{X}$ of all completely linear functionals (([2]), p. 239), called the **Nakano conjugate** of X.

A **KN-space** is a K-space X which is simultaneously a normed space in which the norm is monotone, i.e., from $|x| \le |y|$ it follows that $\|x\|_X \le \|y\|_X$.

A **KB-space** is a KN-space X in which two additional conditions are satisfied:

(A) If $x_n \downarrow 0$, then $\|x_n\|_X \to 0$.

(B) If $0 \le x_n \uparrow$ and $\lim \|x_n\|_X < \infty$, then there exists $\sup x_n \in X$.

For an arbitrary KN-space X, by X^* we denote its Banach dual. Recall that $X^* \subset \widehat{X}$, and, if X is a Banach space, then $X^* = \widehat{X}$.

§ 1. Let Q be an extremally disconnected bicompact space, $W = C_\infty(Q)$ the corresponding extended K-space. For brevity we shall denote $C(Q)$, i.e., the ordinary space of real finite continuous functions on Q, by M.

Definition 1. Let X be a normal subspace in $C_\infty(Q)$, $f \in \widehat{X}$, $u \in X_+$. For any $x \in M$ put

$$f_{(u)}(x) = f(xu), \tag{1}$$

where xu is the product in the sense of multiplication in $C_\infty(Q)$, see ([2]), p. 163. It is clear that $f_{(u)} \in \widehat{M}$.

Definition 2. Let X and Y be normal subspaces in $C_\infty(Q)$, $f \in \widehat{X}$, $g \in \widehat{Y}$. We shall say that f and g are **disjoint** (notation fDg) if, for any $u \in X_+$, $v \in Y_+$, $f_{(u)} d g_{(v)}$ holds, i.e., $f_{(u)} g_{(v)}$ are disjoint as elements of the K-space $\widehat{M}$.

We emphasize that one cannot speak about the disjointness of the elements f, g in the ordinary sense, since they are not elements of one and the same K-space.

Theorem 1. Let X be a normal subspace in $C_\infty(Q)$. Fix a unit 1_X in the space $\mathfrak{M}(\widehat{X})$ and a unit 1_M in $\mathfrak{M}(\widehat{M})$. Then there exists a unique pair (R_X, V_X), where V_X is a component in $\mathfrak{M}(\widehat{M})$, and R_X is an isomorphism of the K-space $\mathfrak{M}(\widehat{X})$ onto the K-space V_X, satisfying the conditions:

(1) For any $f \in \widehat{X}$, $g \in \widehat{M}$,

$$(fDg) \Longleftrightarrow (R_X f \, dg);$$

(2) $R_X(1_X) = \mathrm{Pr}_{V_X} 1_M$.

Let us note that here $R_X f$ and g are elements of one and the same K-space $\mathfrak{M}(\widehat{M})$, and one may speak of their disjointness in the ordinary sense.

Definition 3. The operator R_X introduced in Theorem 1 will be called the **canonical realization of the space** X.

It is clear that the operator R_X depends on the choice of the units $1_X, 1_M$ in the spaces $\mathfrak{M}(\widehat{X})$, $\mathfrak{M}(\widehat{M})$, respectively.

Theorem 2. Let X and Y be normal subspaces in $C_\infty(Q)$; R_X and R_Y the corresponding canonical realizations. Then, for any $f \in \widehat{X}$, $g \in \widehat{Y}$, and for any choice of the units $1_M, 1_X, 1_Y$, one has

$$(fDg) \Longleftrightarrow (R_X f\, dR_Y g).$$

§ 2. Throughout this section we assume that a unit has been chosen in $\mathfrak{M}(\widehat{M})$ and that a realization $\mathfrak{M}(\widehat{M}) = C_\infty(Q')$ has been made on a suitable extremally disconnected bicompact space Q'. Let X_0, X_1 be Banach KN-spaces that are normal subspaces in $C_\infty(Q)$. Following A. P. Calderón ([3]), put, for $0 < s < 1$,

$$X_0^{1-s}X_1^s = \{z \in C_\infty(Q) : |z| \le \lambda x_0^{1-s} x_1^s, \text{ where } 0 \le x_i \in X_i,$$
$$\|x_i\|_{X_i} \le 1 \ (i = 0, 1), \text{ the number } \lambda > 0\}, \tag{2}$$

and for $z \in X_0^{1-s}X_1^s$ we take as $\|z\|_{X_0^{1-s}X_1^s}$ the infimum of all possible λ in (2). Then $(X_0^{1-s}X_1^s, \|\cdot\|_{X_0^{1-s}X_1^s})$ is a Banach KN-space.

Let us for the moment choose units arbitrarily in the spaces $\mathfrak{M}(X_0^*)$, $\mathfrak{M}(X_1^*)$, $\mathfrak{M}((X_0^{1-s}X_1^s)^*)$, and identify the spaces themselves X_0^*, X_1^*, $(X_0^{1-s}X_1^s)^*$ with their images in $C_\infty(Q')$ under the canonical realizations. After this one may consider the Calderón space $(X_0^*)^{1-s}(X_1^*)^s$, constructed from X_0^* and X_1^* in the same way (formula (2)) in which the space $X_0^{1-s}X_1^s$ is constructed from X_0 and X_1.

Theorem 3. *Let the units in the spaces $\mathfrak{M}(X_0^*)$ and $\mathfrak{M}(X_1^*)$ be chosen arbitrarily. Then in the space $\mathfrak{M}((X_0^{1-s}X_1^s)^*)$ one can choose a unit so that, when the corresponding spaces are identified with their images under the canonical realizations, the equality*

$$(X_0^{1-s}X_1^s)^* = (X_0^*)^{1-s}(X_1^*)^s, \tag{3}$$

will hold both as to the stock of elements and as to the norm.

The **proof** of this theorem is based on results previously obtained by the author ([4,5]).

It follows from Theorem 3 that the Banach conjugates to the family $X_0^{1-s}X_1^s$ $(0 < s < 1)$ again form a similar family. We emphasize here that in Theorem 3 no additional restrictions are imposed on the Banach KN-spaces X_0 and X_1.

Let us now consider an important special case of Calderón’ s construction. Let X be a Banach KN-space that is a normal subspace in $C_\infty(Q)$; let $p > 1$ be an arbitrary number. Put

$$X_p = \{x \in C_\infty(Q) : \ |x|^p \in X\} \tag{4}$$

and, for $x \in X_p$,

$$\|x\|_{X_p} = \||x|^p\|_X^{1/p}. \tag{5}$$

It is clear that $X_p = X^{1-s}Y^s$, where $Y = C(Q)$ and $1 - s = 1/p$.

Theorem 4. a) *The formula $(X_p)^* = (X^*)_p$ is valid, where X^* is the Nakano conjugate to the Banach conjugate X^*.*

b) *The Banach conjugate of odd order to X_p is a KB-space.*

c) *If X is not a KB-space, then no Banach conjugate of even order to X_p is a KB-space.*

Theorem 5. *Let $\overline{X}$ be total on X and let the following condition be satisfied: if a directed system $0 \le x_\alpha \uparrow$ $(\alpha \in A)$ and $\sup \|x_\alpha\|_X < \infty$, then there exists $\sup x_\alpha \in X$ and $\sup \|x_\alpha\|_X = \|\sup x_\alpha\|_X$.*

Then X_p is algebraically and structurally isomorphic and isometric to $(\overline{X}_p)^*$.

Using Theorem 5, some other results of the author ([5]), and the Bishop–Phelps theorem on support functionals ([1]), one can prove the following theorem on completely linear functionals in a KN-space supporting its unit ball.

Theorem 6. *Let X be a Banach KN-space satisfying all the conditions of Theorem 5. Then:*

a) *for any $x \in X$ and any number $\varepsilon > 0$ there exist $y \in X$ and $f \in \overline{X}$ such that $\|x - y\|_X < \varepsilon$, $\|f\|_{X^*} = 1$ and $f(y) = \|y\|_X$;*

b) *for any $f \in \overline{X}$ and any number $\varepsilon > 0$ there exist $g \in \overline{X}$ and $x \in X$ such that $\|f - g\|_{X^*} < \varepsilon$, $\|x\|_X = 1$ and $g(x) = \|g\|_{X^*}$;*

c) *if $\mathfrak{M}(X)$ is of countable type, then there exist a weak unit 1 in X and a functional $f \in \overline{X}$ such that $\|1\|_X = \|f\|_{X^*} = f(1) = 1$.*

In conclusion the author expresses gratitude to Prof. B. Z. Vulikh for his attention to the present work.

Received
1 II 1969

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Note: Figure translations are in progress. See original paper for figures.