SUBGROUPS, NORMAL DIVISORS, AND EPIMORPHIC IMAGES OF A FINITE \(p\)-GROUP

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Abstract

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SUBGROUPS, NORMAL DIVISORS, AND EPIMORPHIC IMAGES OF A FINITE p-GROUP

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1. After the appearance of P. Hall' s fundamental work on the theory of p-groups ([1]), a number of important results were obtained. These are, first of all, the results of N. Blackburn (p-groups of maximal class and p-groups with small abelian normal divisors), A. I. Kostrikin (the solution of the weakened Burnside problem for prime exponent), E. S. Golod and I. R. Shafarevich (the study of relations between the number of generators and defining relations, which led to a negative solution of the fundamentally important problem on the class field tower), and V. Ganong (a proof of the existence of outer automorphisms of a p-group).

In recent years interest has revived in counting the number of subgroups of a given order and of a given structure in a p-group G. Knowledge of this number makes it possible to obtain an assertion on the existence of a normal divisor with a given property in G: such a normal divisor certainly exists if the indicated number is not divisible by p. Proceeding along this path, it was possible to obtain a fundamental strengthening of Blackburn' s important theorem on p-groups of odd order without normal subgroups of type (p, p, p) (see Theorem 1 below). The advantage of such an approach is illustrated especially clearly by the corollary to Theorem 1. Theorem 2 is a very nontrivial consequence of the preceding one; it generalizes another theorem of Blackburn (see Theorem 3.11.12 in ([2])). For the proof of Theorem 1 a lemma is needed which improves a well-known result of J. L. Alperin on centralizers of abelian normal divisors ([3]).

In Theorem 3 a number of results are collected on the number of subgroups of a given order and of a given structure. From parts c) and d) of this theorem there follows the principal result of Theorem 1 for $p > 3$ (for smaller values of p, separate arguments are carried out).

Theorem 5, in a certain sense dual to Theorem 5.4 of ([5]), may prove useful for computing the Schur multiplier of a metacyclic p-group, and also in the study of the normal structure of a p-group (cf. also Theorem 6 and Theorem 5.1 of ([5])).

Many theorems on the number of subgroups of a given order and of a given structure can be found in ([4–6]). A new theorem on the number of subgroups of order p, $p > 3$, was proved by B. Huppert (Theorem 3.11.8 in ([2])); independently and simultaneously this result was proved in ([5]) (see Corollary 5.5 there). Recently J. Thompson showed that in a noncyclic 2-group which is not a group of maximal class, the number of solutions of the equation $x^2 = 1$ is divisible by 4 (Theorem 6.2 in ([7])). It is easy to see that this result of Thompson is a special case of Corollary 5.3 of ([5]) (this corollary was proved without using representation theory, as in Thompson).

In what follows, unless the contrary is stated, only finite p-groups are considered, and p is always a prime number. The standard notation and concepts used below are not explained; the less well-known ones are given below.

$p^{d(G)} = |G : \Phi(G)|$; $\Omega_1(G) = \langle x \mid x^p = 1\rangle$; $m(G) = p - 1$, if G is irregular in the sense of Hall; otherwise $m(G) = |\Omega_1(G)|$; $\exp G$ is the exponent of G (the least common multiple of the orders of all elements of the group G); $\varphi_{a,b}$ is the number of subgroups of order p^b in an elementary abelian group of order p^a; a Blackburn group is a p-group without subgroups of type

(p, p, p); a Dedekind group is a group in which all subgroups are normal; a reduced group is a non-Dedekind p-group whose commutator subgroup is its only normal divisor of order p; a G-admissible subgroup is a subgroup normal in G.

2. Blackburn groups have been completely studied in ([8,9]) for odd p (see also Chapter 3 in ([2])).

Theorem 1. *Let the p-group G not be a Blackburn group; moreover, if $p = 2$, we additionally require that G contain no subgroup isomorphic to the direct product of a group of order 2 and a dihedral group of order 8. Then the number of Blackburn subgroups of order p^n, $n > 2$, contained in G, is divisible by p.*

Corollary. *Let G be a normal subgroup of a p-group H; let G be as in Theorem 1. Then G contains an H-admissible subgroup of type (p, p, p).*

The structure of 2-groups of Blackburn has not yet been studied; their description under a strong additional assumption was given by Thompson (see Chapter 3 in ([2])).

Theorem 2. *Let N be a normal subgroup of a p-group G, p odd, $|N| = p^n$, $3 < r < n - 1$. If all G-admissible subgroups of order p^r in N are generated by two elements, then N is either metacyclic or a 3-group of maximal class. The converse is also true.*

For the proof of Theorem 1, the following lemma, of independent interest, is useful.

Lemma. *Let M be a normal subgroup of a p-group G, $A \subset B \subseteq M$, where A and B are abelian, A is normal in G,* $\exp A = \exp B = p^n > 2$. *Then in M there exists an abelian G-admissible subgroup D such that $A \subset D$ and* $\exp D = \exp A$ *(A is properly contained in B and in D).*

3. For the proof of Theorem 3, the theory of regular p-groups developed in ([1,4]), as well as the sufficient criteria of regularity given there, is essentially used.

Theorem 3. *In a p-group G the following assertions hold:*

a) G contains a number congruent to 1 modulo p of subgroups of exponent p and order $p^{m(G)}$.

b) If $m(G) > 1$, then G contains a number congruent to $1 + p$ modulo p^2 of subgroups of order $p^{m(G)-1}$ and exponent p.

c) If $2 < r < m(G)$, then G contains a number divisible by p of such subgroups H of order p^r that $\exp H = p$ and $d(H) = 2$.

d) If $0 < k < m(G)$, $r \geq k$, and H is a subgroup normal in G such that $|H| = p^r$ and $m(H) = k$, and n is a fixed natural number, then between H and G there is contained a number divisible by p of such subgroups F that $|F| = p^{r+n}$ and $m(F) = k$. If $n = 1$, then the assumption on the normality of H may be omitted (in what follows, such remarks will no longer be made).

e) If $1 < r < m(G) - 1$, then G contains a number congruent to $1 + p + 2p^2$ modulo p^3 of subgroups of order p^r and exponent p.

f) Let H be a normal cyclic subgroup of order p^n in G; let the natural number k be such that $k > 1$, if $H = 1$, and $k = n+1$ otherwise. Then the number of cyclic subgroups of order p^k contained between H and G is divisible by $p^{m(G)-1}$.

g) The number of subgroups of order p in G is congruent to $\varphi_{m(G),1}$ modulo $p^{m(G)}$.

Theorem 4. *Let the 2-group G be noncyclic and not a group of maximal class; let N be a normal subgroup in G; and let n be a fixed natural number. Then the following assertions hold:*

a) If $2^n > \max\{4, |N|\}$, then between N and G there is contained an even number of subgroups of maximal class and of order 2^n; this number is even divisible by 4 if $N = \Phi(G)$.

b) If $2^n > \max\{2, |N|\}$, then between N and G there is contained an even number of cyclic subgroups of order 2^n.

4. We regard two epimorphisms of a group as identical if and only if their kernels coincide.

Theorem 5. Let G be a nonmetacyclic p-group; let n be a natural number such that $n > 2$ if p is odd, and $n > 3$ if $p = 2$. The number of distinct epimorphisms of the group G onto metacyclic groups of order p^n is always divisible by p.

In particular, if all epimorphic images of the group G are metacyclic, then either G itself is metacyclic, or $|G| \leq p^4$ (if p is odd, even $|G| = p^3$). Recall that a group of maximal class is called a nonabelian group of order p^n and class $n-1$. Although 2-groups of maximal class are metacyclic, Theorem 6 does not follow from Theorem 5.

Theorem 6. Let the 2-group G not be a group of maximal class, $n > 2$. Then the number of distinct epimorphisms of G onto groups of maximal class and order 2^n is always even.

5. In Theorem 7 a construction is given of a "universal" quotient group of a non-Dedekind p-group. Its main part is a reduced group.

Theorem 7. Let H be a normal subgroup of a non-Dedekind p-group G, let the quotient group G/H be abelian, and moreover let the quotient group G by any nonidentity G-admissible subgroup of H be Dedekind. Then the commutator subgroup of G is the unique G-admissible subgroup of order p in H. If $H = \Phi(G)$, then $G = A \times R$, where A is an elementary abelian subgroup and R is a reduced subgroup.

All noninvariant subgroups of a reduced p-group are abelian of exponent p. A broader class of groups is described in Theorem 8 (in this theorem the finiteness of the group G is not assumed).

Theorem 8. Let in a non-Dedekind periodic group G every noninvariant subgroup have prime exponent. If G has an element of composite order, then at least one of the following assertions is true:

a) G is a p-group of class 2.

b) G is the dihedral group of order 16.

c) G is a Frobenius group of order p^2q with an invariant cyclic subgroup.

d) $G = A \times B$, where A is an abelian group of exponent p, and B is a nonabelian group of order pq, with prime number $q \equiv 1 \pmod p$.

Let us note that in the proof of Theorem 1 the results of Blackburn on p-groups without normal subgroups of type (p, p, p) are not used. The author has a proof of Theorem 1 that is uniform for all p. In the proof of Theorem 2, the characterization, given by Blackburn, of 3-groups of maximal class is used at one point. For the proof of Theorem 6 the principle of enumerating normal divisors, in a certain sense dual to Hall' s enumeration principle, proved very useful. Theorem 8 improves one result of A. Mann, who studied p-groups in which every noninvariant subgroup is abelian of exponent p. We note that nonabelian groups of exponent p with this property either have commutator subgroup of order p, or have order p^4 and nilpotency class 3 (and conversely).

Let us note that the Frattini subgroup of a reduced group lies in the center and therefore is cyclic; groups with cyclic Frattini subgroup are described in ([6]).

Let us note that many results of Theorem 3 follow from item a) of this theorem. Almost everywhere in this theorem the number p must be sufficiently large (in particular, for 2-groups this theorem is trivial and gives nothing new in comparison with known results for 3-groups, if one does not count items c)–d) and partly e)).

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Note: Figure translations are in progress. See original paper for figures.