THE GREEN MATRIX OF AN INHOMOGENEOUS PARABOLIC BOUNDARY-VALUE PROBLEM

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S. D. IVASISHEN

THE GREEN MATRIX OF AN INHOMOGENEOUS PARABOLIC BOUNDARY-VALUE PROBLEM

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In the works of S. D. Eidelman and the author ($^{1-3}$) the Green matrix of the general homogeneous boundary-value problem for a parabolic system in the sense of I. G. Petrovskii was studied. Here an investigation is carried out in the general case of the Green matrix of an inhomogeneous parabolic boundary-value problem. In doing so, methods developed by Yu. P. Krasovskii (^{4}) in the elliptic case, by V. A. Solonnikov (^{5}), and by S. D. Eidelman and the author ($^{1-3,6}$) in the study of parabolic boundary-value problems are used.

We note that the Green matrix of the general elliptic boundary-value problem has been subjected to detailed investigation in the works of Yu. M. Berezanskii and Ya. A. Roitberg (^{7,8}), Yu. P. Krasovskii (^{4}), and others.

1. In the domain $Q_0 = [0, T] \times \Omega_0$, where T is a positive number and Ω_0 is a bounded domain with boundary Ω_1 in the space E_n, consider a system of N equations containing N unknown functions,

$$L(t, x, D_t, D_x)u \equiv D_{tu} - \sum_{|k|\le 2b} a_k(t, x) D_x^k u = 0, \tag{1}$$

the initial condition

$$u|_{t=0} = 0 \tag{2}$$

and the boundary conditions

$$B_j(t, x, D_x)u|_{Q_1} \equiv \sum_{|k|\le r_j} b_{jk}(t, x) D_x^k u|_{Q_1} = f_j(t, x) \quad (j = 1, \dots, bN), \tag{3}$$

where $Q_1 = [0, T] \times \Omega_1$, $r_j \le 2b - 1$.

We shall denote by $H_i^{l+\alpha}$ $(i = 0, 1)$ the spaces $C_{x,t}^{l+\alpha,\,(l+\alpha)/2b}(Q_i)$, defined in $(^5)$, and by $\overset{\circ}{H}{}_i^{l+\alpha}$ and $\overset{*}{H}{}_i^{l+\alpha}$ the sets of functions $f(t,x)$ from $H_i^{l+\alpha}$ satisfying, respectively, the conditions

$$D_t^k f|_{t=0} = 0, \qquad D_t^k f|_{t=T} = 0 \quad \text{for } k = 0, 1, \dots, [l/2b].$$

With respect to problem (1)–(3) assume that: (A) it is parabolic, i.e. system (1) is parabolic in the sense of I. G. Petrovskii and the operators B_j are connected with system (1) by the complementing condition; (B_l) the boundary Ω_1 belongs to the class $C^{2b+l+\alpha}$ in the sense of $(^5)$, and the coefficients $a_k(t,x) \in H_0^{l+\alpha}$, $b_{jk}(t,x) \in H_1^{l+\alpha}$, where l is some positive integer and $0 < \alpha < 1$.

It is known $(^5)$ that if condition (A) and the condition (B'_m), differing from (B_m) only in that the requirement $b_{jk} \in H_1^{m+\alpha}$ is replaced by $b_{jk} \in H_1^{2b-r_j+m+\alpha}$, are fulfilled, then for any $f_j(t,x) \in \overset{*}{H}{}_1^{2b-r_j+m+\alpha}$ there exists a unique solution of problem (1)–(3) belonging to $\overset{*}{H}{}_0^{2b+m+\alpha}$, where m is any nonnegative integer.

In the present work it is proved that, under conditions (A),

(B_l) with sufficiently large l, this solution is representable in the form

$$u(t,x) = \sum_{j=1}^{bN} \int_0^t d\tau \int_{\Omega_1} G_j(t,x;\tau,\xi) f_j(\tau,\xi)\, d\xi,$$

where $d\xi$ is the surface element of Ω_1. The matrix $G(t,x;\tau,\xi)$, whose columns are the vector-functions $G_j(t,x;\tau,\xi)$, will be called the **Green matrix** of problem (1)–(3). In this paper the matrix G is constructed as the sum of the principal part with respect to the order of the singularity and a remainder, and exact estimates of its derivatives in the closed domain are established.

We define the principal part $G^{(0)}(t,x;\tau,\xi)$ of the Green matrix G. Let (τ,ξ) be an arbitrarily fixed point of Q_1, and let $\nu(\xi)$ be the unit vector of the inward normal to Ω_1 at the point ξ. Put

$$G^{(0)}(t,x;\tau,\xi) = \chi(x-\xi)\widetilde{G}^{(0)}(t-\tau,z;\tau,\xi), \tag{4}$$

$$z = x^*(x) - \xi - ((x^*(x) - \xi), \nu(\xi))\nu(\xi) + \rho(x)\nu(\xi),$$

where $\chi(x)$ is an infinitely differentiable function equal to zero for $|x| \geq d/2$ and to 1 for $|x| \leq d/3$; d is the radius of the Lyapunov sphere of the surface Ω_1; $x^*(x)$ is the point of Ω_1 nearest to x; $\rho(x)$ is the distance from the point x to Ω_1, and $\widetilde{G}^{(0)}(t-\tau,z;\tau,\xi)$ is the half-space fundamental matrix of solutions of the problem $(5,6)$

$$D_t u - \sum_{|k|=2b} a_k(\tau,\xi) D_z^k u = 0,$$

$$u\big|_{t=\tau} = 0,$$

$$\sum_{|k|=r_j} b_{jk}(\tau,\xi) D_z^k u\big|_{(z,\nu(\xi))=0} = f_j(t,z) \qquad (j = 1, \ldots, bN)$$

in the domain $t > \tau$, $(z,\nu(\xi)) > 0$.

2. We introduce here some definitions. Let p, r, s, i, j be integers, with $r, s \geq 0$, while i, j can take only the values 0 or 1. Denote by $U^p_{r+\alpha,s+\alpha}(Q_i, Q_j)$ the class of vector-functions $K(t,x;\tau,\xi)$ which are defined for $(t,x) \in Q_i$, $(\tau,\xi) \in Q_j$, $(t,x) \neq (\tau,\xi)$, are such that $K(t,x;\tau,\xi) = 0$ for $t < \tau$, and have derivatives of the form $D_t^{k_0} D_x^k D_\tau^{m_0} D_\xi^m K(t,x;\tau,\xi) = K^{(\bar{k},\bar{m})}$, $\bar{k} = (k_0, k)$, $\bar{m} = (m_0, m)$, $|\bar{k}| = 2bk_0 + |k| \leq r$, $|\bar{m}| = 2bm_0 + |m| \leq s$, satisfying the following inequalities:

$$|K^{(\bar{k},\bar{m})}| \leq C\,[d^{-n-2b+p-|\bar{k}|-|\bar{m}|}(t,x;\tau,\xi) + 1]\,\Phi_c(t,x;\tau,\xi;t-\tau),$$

$$|\Delta_{x_1}^{x_2} K^{(\bar{k},\bar{m})}| \leq C|x_1 - x_2|^\alpha [d^{-n-2b+p-r-|\bar{m}|-\alpha}(t,x^*;\tau,\xi) + 1]\times$$

$$\times\Phi_c(t,x^*;\tau,\xi;t-\tau)$$

$$(|\bar{k}| = r,\ |x^* - \xi| = \min_{i=1,2} |x_i - \xi|),$$

$$|\Delta_{t_1}^{t_2} K^{(\bar{k},\bar{m})}| \leq C(t_1 - t_2)^{(r-|\bar{k}|+\alpha)/2b}[d^{-n-2b+p-r-|\bar{m}|-\alpha}(t_2,x;\tau,\xi) + 1]\times$$

$$\times\Phi_c(t_2,x;\tau,\xi;t_1-\tau)$$

$$(r - 2b < |\bar{k}| \leq r,\ \tau < t_2 < t_1),$$

$$|\Delta_{\xi_1}^{\xi_2} K^{(\bar{k},\bar{m})}| \leq C|\xi_1 - \xi_2|^\alpha [d^{-n-2b+p-|\bar{k}|-s-\alpha}(t,x;\tau,\xi^*) + 1]\times$$

$$\times\Phi_c(t,x;\tau,\xi^*;t-\tau)$$

$$(|\bar{m}| = s,\ |x-\xi^*| = \min_{i=1,2}|x-\xi_i|),$$

$$|\Delta_{\tau_1}^{\tau_2} K^{(\bar{k},\bar{m})}| \le C(\tau_1-\tau_2)^{(s-|\bar{m}|+\alpha)/2b}[d^{-n-2b+p-|\bar{k}|-s-\alpha}(t,x;\tau_1,\xi)+1]\times \tag{5}$$

$$\times\Phi_c(t,x;\tau_1,\xi;t-\tau_2)$$

$$(s-2b<|\bar{m}|\le s,\ \tau_2<\tau_1<t),$$

$$\left|\Delta_{x_1}^{x_2}\Delta_{\xi_1}^{\xi_2}K(\bar{k},\bar{m})\right| \le C|x_1-x_2|^{\alpha}|\xi_1-\xi_2|^{\alpha}\left[d^{-n-2b+\gamma-r-s-2\alpha}(t,x^*;\tau,\xi^*)+1\right]\times$$

$$\times\Phi_c(t,x^*;\tau,\xi^*;t-\tau)$$

$$\left(|\bar{k}|=r,\ |\bar{m}|=s,\ |x^*-\xi^*|=\min_{i,j=1,2}|x_i-\xi_j|\right),$$

$$\left|\Delta_{t_1}^{t_2}\Delta_{\xi_1}^{\xi_2}K(\bar{k},\bar{m})\right| \le C(t_1-t_2)^{(r-|\bar{k}|+\alpha)/2b}|\xi_1-\xi_2|^{\alpha}\times$$

$$\times\left[d^{-n-2b+p-r-s-2\alpha}(t_2,x;\tau,\xi^*)+1\right]\Phi_c(t_2,x;\tau,\xi^*;t_1-\tau)$$

$$\left(r-2b<|\bar{k}|\le r,\ \tau<t_2<t_1,\ |x-\xi^*|=\min_{i=1,2}|x-\xi_i|\right),$$

$$\left|\Delta_{x_1}^{x_2}\Delta_{\tau_1}^{\tau_2}K(\bar{k},\bar{m})\right| \le C|x_1-x_2|^{\alpha}(\tau_1-\tau_2)^{(s-|\bar{m}|+\alpha)/2b}\times$$

$$\times\left[d^{-n-2b+p-r-s-2\alpha}(t,x^*;\tau_1,\xi)+1\right]\Phi_c(t,x^*;\tau_1,\xi;t-\tau_2)$$

$$\left(s-2b<|\bar{m}|\le s,\ \tau_2<\tau_1<t,\ |x^*-\xi|=\min_{i=1,2}|x_i-\xi|\right),$$

$$\left|\Delta_{t_1}^{t_2}\Delta_{\tau_1}^{\tau_2}K(\bar{k},\bar{m})\right| \le C(t_1-t_2)^{(r-|\bar{k}|+\alpha)/2b}(\tau_1-\tau_2)^{(s-|\bar{m}|+\alpha)/2b}\times$$

$$\times\left[d^{-n-2b+p-r-s-2\alpha}(t_2,x;\tau_1,\xi)+1\right]\Phi_c(t_2,x;\tau_1,\xi;t_1-\tau_2)$$

$$\left(r-2b<|\bar{k}|\le r,\ s-2b<|\bar{m}|\le s,\ \tau_2<\tau_1<t_2<t_1\right),$$

where

$$\Phi_c(t,x;\tau,\xi;t_1-\tau_2)=\exp\left\{-c\left[\frac{d(t,x;\tau,\xi)}{(t_1-\tau_2)^{1/2b}}\right]^{2b/(2b-1)}\right\};$$

$$d(t,x;\tau,\xi)=\sqrt{(t-\tau)^{1/b}+|x-\xi|^2};$$

$\Delta_{x_1}^{x_2}$, $\Delta_{\xi_1}^{\xi_2}$, $\Delta_{t_1}^{t_2}$, $\Delta_{\tau_1}^{\tau_2}$ are operations of taking differences with respect to x, ξ, t, and τ, respectively. If K is defined, for example, with respect to x on Ω_1, then by D_x^k here is meant some derivative of order $|k|$ with respect to $\bar{x}_1, \dots, \bar{x}_{n-1}$, where $\bar{x}_1, \dots, \bar{x}_{n-1}$ is a local coordinate system on Ω_1.

Let A be a linear operator acting from $H_j^{l+\alpha}$ into $H_i^{l'+\alpha}$. We shall say that the operator A has kernel $K(t, x; \tau, \xi)$ if there exists a function $K(t, x; \tau, \xi)$, defined for $(t, x) \in Q_i$, $(\tau, \xi) \in Q_j$, $(t, x) \neq (\tau, \xi)$, equal to zero for $t < \tau$, and such that

$$(Af)(t, x) = \int_0^t d\tau \int_{\Omega_j} K(t, x; \tau, \xi) f(\tau, \xi)\, d\xi$$

for every point $(t, x) \in Q_i$ and arbitrary function $f \in H_j^{l+\alpha}$ which vanishes in a neighborhood of the point (t, x), if $(t, x) \in Q_j$.

Linear operators A and A^*, acting respectively from $H_j^{l+\alpha}$ and $H_i^{l'+\alpha}$ into $H_i^{l'+\alpha}$ and $H_j^{l+\alpha}$, will be called adjoint if for any functions $f \in H_j^{l+\alpha}$ and $g \in H_i^{l'+\alpha}$ the equality

$$\int_0^T dt \int_{\Omega_i} (Af)(t, x) g(t, x)\, dx = \int_0^T d\tau \int_{\Omega_j} f(\tau, \xi)(A^* g)(\tau, \xi)\, d\xi$$

holds.

The following lemma holds; it is the parabolic analogue of the lemma on the composition of kernels with nonintegrable singularities from (4) and plays an essential role in the proof of the main result of the paper.

Lemma. Let the operators A, B, and B^* act boundedly, respectively, from $H_m^{r-p+m+\alpha}$, $H_j^{k+\alpha}$, and $\dot{H}_m^{s'-t'+m+\alpha}$ into $H_i^{r+\alpha}$, $H^{r-p+m+\alpha}$, and $H_j^{s'+\alpha}$, where $r - p + m \geq 0$, $s' - p' + m \geq 0$. Suppose, further, that the operator A has kernel $K(t, x; \tau, \xi) \in U^{p}_{r+\alpha,\, s'-p'+m+\alpha}(Q_i, Q_m)$, and the operator B has kernel $K'(t, x; \tau, \xi) \in U^{p'}_{r-p-m+\alpha,\, s'+\alpha}(Q_m, Q_j)$. Then the operator AB has kernel $K''(t, x; \tau, \xi) \in U^{p+p'-m}_{r+\alpha,\, s'+\alpha}(Q_i, Q_j)$.

3. The following main theorem is valid.

Theorem. Suppose conditions (A) and (B_1) are satisfied with $l \geq \max(2l_0 - 1, l_0 + 1)$, where $l_0 = \max_j (2b - r_j)$; then there exists a Green matrix $G(t, x; \tau, \xi)$ for problem (1)–(3), whose j-th column $G_j(t, x; \tau, \xi)$ has the following properties: 1) it belongs to the class

$$U^{r_j+1}_{2b+l-l_0+\alpha,\, l-2b+r_j+1+\alpha}(Q_0, Q_1),$$

and the constants C in estimates (5) for G_j depend on the corresponding norms of the coefficients of system (1) and of the boundary conditions (3), on the various characteristics of Ω_0 and Ω_1, on the numbers $n, N, b, r_j, l, \alpha, T$, and on the constants in the parabolicity and complementarity conditions; 2) there is a representation

$$G_j(t, x; \tau, \xi) = G_j^{(t)}(t, x; \tau, \xi) + v_j(t, x; \tau, \xi),$$

where $G_j^{(t)}$ is the j-th column of matrix (4), and

$$v_j \in U^{r_j+2}_{2b+l-l_0+\alpha,\, l-2b+r_j+1+\alpha}(Q_0, Q_1);$$

3) for $t > \tau$ the equalities

$$L(t, x, D_t, D_x) G_j(t, x; \tau, \xi) = 0, \qquad B_i(t, x, D_x) G_j(t, x; \tau, \xi)\big|_{Q_1} = 0$$
$$(i, j = 1, \dots, bN)$$

hold.

The proof of the theorem is carried out according to the scheme proposed in the elliptic case by Yu. P. Krasovskii (^{4}); at the same time, the technique developed in the theory of parabolic boundary-value problems (the study of parabolic potential kernels) is systematically used.

Remark. Denote by $G_0(t, x; \tau, \xi)$ the Green matrix of the homogeneous boundary-value problem

$$L(t, x, D_t, D_x) u = f(t, x), \tag{6}$$

$$u\big|_{t=0} = 0, \qquad B_j(t, x, D_x) u\big|_{Q_1} = 0 \qquad (j = 1, \dots, bN),$$

constructed in (^{1}). With the aid of the method used in the proof of the theorem, one can prove that, under the assumptions of this theorem, the columns of the matrix G_0 belong to the class $U^{2b}_{b+l-l_0+\alpha,\, l+\alpha}(Q_0, Q_0)$.

With the aid of the matrices G_0 and G, the solution of problem (6), (2), (3), belonging to $H_0^{2b+m+\alpha}$ $(m \le l - l_0)$, is given by the equality

$$u(t, x) = \int_0^t d\tau \int_{\Omega_0} G_0(t, x; \tau, \xi) f(\tau, \xi)\, d\xi + \sum_{j=1}^{bN} \int_0^t d\tau \int_{\Omega_1} G_j(t, x; \tau, \xi) f_j(\tau, \xi)\, d\xi.$$

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Chernivtsi State
University

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Note: Figure translations are in progress. See original paper for figures.

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