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Abstract

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ON AN ASYMPTOTIC METHOD FOR SOLVING THE EQUATIONS OF BROWNIAN MOTION OF AEROSOL PARTICLES

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Let a plane body—an obstacle with a sufficiently smooth surface Γ—be located in a flow of a gaseous medium. Construct an orthogonal curvilinear coordinate system $O\xi\eta$ as follows: place the origin in the front part of Γ, direct the axis $O\eta$ along Γ, and the axis $O\xi$ perpendicular to Γ. The equations of Brownian motion of aerosol particles, obtained in ([1]), upon introducing the convective velocity $\mathbf{v}^* = \mathbf{v} + \frac{1}{\mathrm{Pe}}\frac{\partial \ln n}{\partial r}$, in the coordinate system $O\xi\eta$ take the form:

$$\frac{\partial n}{\partial t}+\frac{\partial n v_\xi^*}{\partial \xi}+\frac{1}{R+\xi}n v_\xi^*+\frac{R}{R+\xi}\frac{\partial n v_\eta^*}{\partial \eta} = \frac{1}{\mathrm{Pe}}\frac{R}{R+\xi}\left\{\frac{\partial}{\partial \xi}\left(1+\frac{\xi}{R}\right)\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\frac{R}{R+\xi}\frac{\partial}{\partial \eta}\right\}n; \tag{1}$$

$$k\left\{\frac{\partial}{\partial t}+\left(v_\xi^*-\frac{1}{\mathrm{Pe}}\frac{\partial \ln n}{\partial \xi}\right)\frac{\partial}{\partial \xi}+\frac{R}{R+\xi}\left(v_\eta^*-\frac{1}{\mathrm{Pe}}\frac{R}{R+\xi}\frac{\partial \ln n}{\partial \eta}\right)\frac{\partial}{\partial \eta}\right\}\times\left(v_\xi^*-\frac{1}{\mathrm{Pe}}\frac{\partial \ln n}{\partial \eta}\right)-\frac{k}{R+\xi}\left(v_\eta^*-\frac{1}{\mathrm{Pe}}\frac{R}{R+\xi} \right. \tag{2}$$

$$k\left\{\frac{\partial}{\partial t}+\left(v_\xi^*-\frac{1}{\mathrm{Pe}}\frac{\partial \ln n}{\partial \xi}\right)\frac{\partial}{\partial \xi}+\frac{R}{R+\xi}\left(v_\eta^*-\frac{1}{\mathrm{Pe}}\frac{R}{R+\xi}\frac{\partial \ln n}{\partial \eta}\right)\frac{\partial}{\partial \eta}\right\}\times\left(v_\eta^*-\frac{1}{\mathrm{Pe}}\frac{R}{R+\xi}\frac{\partial \ln n}{\partial \eta}\right)+\frac{k}{R+\xi}\left(v_\xi^*-\frac{1}{\mathrm{Pe}} \right. \tag{3}$$

$$n \to n_\infty, \qquad \mathbf{v}_* \to \mathbf{u}+\mathbf{F} \qquad \text{as } r \to \infty.$$

For a fairly broad class of physical problems, the boundary conditions for n on Γ may be written in the form:

Fig. 1

Figure 1: Fig. 1

$$n = 0, \qquad \text{if } v_\xi^*(0,\eta) = 0; \tag{4}$$

$$\frac{1}{\text{Pe}}\frac{\partial n}{\partial \xi} = a(n - n_0), \qquad \text{if } v_\xi^*(0,\eta) \neq 0, \tag{5}$$

where a and n_0 are certain constants determined from particular physical considerations.

In this note the asymptotic behavior of n and $\mathbf{v}^*$ is studied for small k and large Pe.

Let us conditionally divide the flow region into three parts: Q_∞, Q_Γ, and Q_s (see Fig. 1). In Q_∞ the derivatives of n are small; consequently, the solution of equations (1)–(3) can be represented in the form of series

$$n \approx n^{(1)}(\xi,\eta;k;\text{Pe}) = \sum_{m\geq 0} n^{m(1)}\text{Pe}^{-m},$$

$$\mathbf{v}^* \approx \mathbf{v}^{*(1)}(\xi,\eta;k;\text{Pe}) = \sum_{m\geq 0} \mathbf{v}^{(m)(1)}\text{Pe}^{-m}. \tag{6}$$

In Q_Γ, by virtue of the boundary conditions (5), the concentration n in the direction toward Γ must change so substantially that either part of the diffusion terms in (1)–(3) becomes, as $\text{Pe} \to \infty$, of order comparable with the convective terms, or, in the part of Q_Γ adjacent to Q_∞, will be comparable, while in the part of Q_Γ adjacent to Γ, will prevail over the convective terms. Therefore it is necessary first to stretch Q_Γ along the normals to Γ so that the rate of change of the concentration in the direction toward Γ in the transformed region Q_Γ has order $O(1)$, and only after this to seek solutions of equations (1)–(3) in the form of series analogous to (6). It follows from physical considerations that in Q_s the longitudinal changes of concentration will be significant. Since, in principle, the study of the asymptotics in Q_s is analogous to its study in Q_Γ, here we shall restrict ourselves to considering only the regions Q_∞ and Q_Γ.

Fig. 1

We transform the region Q_Γ as follows:

$$\xi = \alpha_0\xi, \quad \eta^* = \eta; \quad \alpha_0 = \alpha_0(\text{Pe}), \quad \alpha_0|_{\text{Pe}\to\infty} \to \infty. \tag{7}$$

Let, in the new variables (we omit the asterisk by η):

$$n \approx n_1 = \tilde{n}(\xi, \eta; k; \mathrm{Pe}),$$

$$\mathbf{v}^* \approx \mathbf{v}_1^* = \alpha_\xi \tilde{v}_{\xi^*}(\xi, \eta; k; \mathrm{Pe})\mathbf{e}_\xi + \alpha_\eta \tilde{v}_\eta^*(\xi, \eta; k; \mathrm{Pe})\mathbf{e}_\eta, \tag{8}$$

$$\alpha_\xi = \alpha_\xi(\mathrm{Pe}), \quad \alpha_\eta = \alpha_\eta(\mathrm{Pe}),$$

$$\tilde{n}, \ \tilde{v}_{\xi^*}, \ \tilde{v}_\eta^* = O(1), \quad \mathrm{Pe} \to \infty,$$

where $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ are the unit vectors of the coordinate system $O\xi\eta$. Suppose that

$$\mathbf{v}^{*(1)} \to O(\xi^{\beta_1})\mathbf{e}_\xi + O(\xi^{\beta_2})\mathbf{e}_\eta, \quad \xi \to 0. \tag{9}$$

Then the condition of asymptotic matching of the functions $\mathbf{v}^{*(1)}$ and $\mathbf{v}_s^*$ on the boundary of the region Q_Γ, and the condition of comparability of the convective and diffusion terms (which determine the variation of the concentration in the direction toward the body), lead to the relations

$$\alpha_0 = \alpha_\xi \mathrm{Pe}, \quad \alpha_\xi = \alpha_0^{-\beta_1}, \quad \alpha_\eta = \alpha_0^{-\beta_2}. \tag{10}$$

Relations (10), for a broad class of physical problems, make it possible to find the principal asymptotic terms for n and $\mathbf{v}^*$ in the region Q_Γ:

$$n_1 \approx \tilde{n}(\xi, \eta; k; \infty), \quad \mathbf{v}^* = \alpha_\xi \tilde{v}_{\xi^*}(\xi, \eta; k; \infty)\mathbf{e}_\xi + \alpha_\eta \tilde{v}_{\eta^*}(\xi, \eta; k; \infty),$$

$$\mathrm{Pe} \to \infty. \tag{11}$$

After this one can construct equations for the functions

$$\tilde{n}(\xi, \eta; k; \mathrm{Pe}) - \tilde{n}(\xi, \eta; k; \infty),$$

$$\tilde{v}_\xi(\xi, \eta; k; \mathrm{Pe}) - \tilde{v}_\xi(\xi, \eta; k; \infty), \tag{12}$$

$$\tilde{v}_\eta(\xi, \eta; k; \mathrm{Pe}) - \tilde{v}_\eta(\xi, \eta; k; \infty),$$

and, analyzing them in an analogous way, obtain the subsequent terms of the asymptotic expansions.

Let us write out the solutions for several concrete problems.

a) Let $\beta_1 = 2$ and $\beta_2 = 1$ (this case is always realized for viscous flow of a medium around an obstacle body and $F = 0$ for $k < k_{\text{cr}}$ ([2],[3]), where k_{cr} is the value of the Stokes number at which inertial deposition begins).

particles on the body). Then, in the stationary case,

$$\alpha_0 = \text{Pe}^{1/3}, \qquad \alpha_\xi = \text{Pe}^{-2/3}, \qquad \alpha_\eta = \text{Pe}^{-1/3},$$

$$\tilde{n}(\zeta, \eta; k; \infty) = \frac{1}{n_\infty} n^{(1)}(0, \eta; k; \infty)\, \tilde{n}(\zeta, \eta; 0; \infty) \times$$

$$\times \exp\left(-2k \int_\eta^\eta \frac{\tilde{u}_\eta}{R}\, d\eta\right) + O(\text{Pe}^{-1/3}),$$

$$\tilde{v}_\xi = \left(\tilde{u}_\xi + \frac{k}{R}\tilde{u}_\eta^2\right)\xi^2 + O(\text{Pe}^{-1/3}),$$

$$\tilde{v}_\eta = \tilde{u}_\eta \xi + O(\text{Pe}^{-1/3}),$$

$$\tilde{n}(\zeta, \eta; 0; \infty) = \frac{n_\infty}{\Gamma(1/3)} \gamma(1/3, 1/9\, \zeta^3 \varphi), \tag{13}$$

$$\varphi = \exp\left[-3 \int^\eta \left(\frac{\tilde{u}_\xi}{\tilde{u}_\eta} + \frac{k}{R}\tilde{u}_\eta\right) d\eta'\right] \left\{\int_{\eta_N}^\eta \tilde{u}_\eta^{-1} \exp\left[-3 \int^{\eta'} \left(\frac{\tilde{u}_\xi}{\tilde{u}_\eta} + \frac{k}{R}\tilde{u}_\eta\right) d\eta''\right] d\eta'\right\}^{-1},$$

$$\tilde{u}_\xi = \zeta^{-2} u_\xi(\zeta \text{Pe}^{-1/3}, \eta) \text{Pe}^{2/3}\big|_{\text{Pe}\to\infty},$$

$$\tilde{u}_\eta = \zeta^{-1} u_\eta(\zeta \text{Pe}^{-1/3}, \eta) \text{Pe}^{1/3}\big|_{\text{Pe}\to\infty},$$

where η_N is the root of the equation $\tilde{u}_\eta(\eta) = 0$. The values of the function $n^{(1)}(0, \eta_N; k; \infty)$ for a sphere in Stokes flow for several k have been calculated in (4). Using these data, one can conclude, on the basis of the formulas given, that for k close to $k_{\text{cr}} = 1.214$, allowance for inertia substantially, by several times, changes the concentration of aerosol particles in the vicinity of the forward critical point of the flow (for $k/k_{\text{cr}} \approx 0.164$, by a factor of 1.85; for $k/k_{\text{cr}} \approx 0.328$, by a factor of 3.01). We note that solution (13) also holds for axisymmetric problems. Analogous solutions in particular form for several specific cases were obtained in (4–7).

b) Let $\beta_1 = 0$ and $\beta_2 = 0$ (this case is realized when a body-obstacle is flowed around by the medium and when there are interaction forces between the body and the aerosol particles). After simple but cumbersome calculations we have:

$$j_\xi = j_1 + O(\mathrm{Pe}^{-1}), \qquad j_1 = -\hat{n}\hat{v}_\xi + \partial\hat{n}/\partial\xi = -\hat{n}\hat{v},$$

$$\tilde{n} = \begin{cases} \hat{n} - a\,\dfrac{\hat{n} - n_0}{a + \hat{u}}\,e^{-\hat{u}\zeta} + O(k\,\mathrm{Pe}), \\ \hat{n}\left(1 + \dfrac{\hat{v} + \hat{u}}{\hat{v}^2}\,\dfrac{\zeta}{k\,\mathrm{Pe}}\right) + (n_0 - \hat{n})\exp\left[-\dfrac{\hat{u}}{\hat{v}^2}\,\dfrac{\zeta}{k\,\mathrm{Pe}}\right] + O\left(\dfrac{1}{k\,\mathrm{Pe}}\right) + O(k), \end{cases} \tag{14}$$

$$\hat{n} = n^{(1)}(0, \eta; k; \infty), \qquad \hat{v} = v_\xi^{*(1)}(0, \eta; k; \infty), \qquad \hat{u} = -u_\xi(0, \eta) - F_\xi(0, \eta).$$

It follows from relations (14) that: 1) the flux of aerosol particles in the direction toward the body, j_ξ, up to terms of order Pe^{-1}, does not depend on ζ and is equal to the flux of aerosol particles obtained when diffusion is completely neglected; 2) the effect of inertia on the motion of aerosol particles in the region Q_Γ may be neglected only for $k \ll \mathrm{Pe}^{-1}$. Comparing formulas (13) and (14), we come to the conclusion that the influence of particle inertia on their Brownian motion is strongest in the case when $u_\xi + F_\xi \neq 0$ on the surface of the body being flowed around and $k \gg \mathrm{Pe}^{-1}$.

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Note: Figure translations are in progress. See original paper for figures.