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Abstract

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CONFORMALLY FLAT GRAVITATIONAL FIELDS CREATED BY DUST-LIKE MATTER

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1. It is known that the metric of any conformally flat space, in a suitable coordinate system, differs from the Cartesian metric of flat space only by a scalar factor. One may expect that this circumstance greatly simplifies the solution of the Einstein equations in conformally flat spaces, since in this case the unknown is a single function (instead of 10 in the general case). However, no one has carried out a systematic investigation of four-dimensional conformally flat spaces as gravitational fields satisfying the Einstein equations. The purpose of the present work is to fill this gap in part.

E. Kasner ([1]) showed that free gravitational fields cannot be conformally flat. But if the energy-momentum tensor $T_{ij} \neq 0$, then the Einstein equations may admit conformally flat solutions. For example, V. A. Fock ([2]) obtained a solution of the Einstein equations for dust-like matter in the form of the metric

$$ds^2 = \left(1 - \frac{A}{\sqrt{x_0^2 - x_1^2 - x_2^2 - x_3^2}}\right)^4 (dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2), \qquad (1)$$

where $A > 0$ is an arbitrary constant. The metric (1) determines the spaces obtained by A. A. Friedmann ([3]) in another coordinate system.

2. We shall consider the question of finding all solutions of the Einstein equations

$$R_{ij} - {}^1/_2 g_{ij} R = -\varkappa \rho u_i u_j, \qquad (2)$$

which lead to metrics of the form

$$ds^2 = H^2(x_0, x_1, x_2, x_3)(dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2). \qquad (3)$$

Substituting (3) into the left-hand side of (2), we obtain

$$V_{ij} + e_i \delta_{ij} W = \varkappa \rho u_i u_j, \qquad (4)$$

where $e_i = \pm 1$ $(e_0 = 1,\ e_1 = e_2 = e_3 = -1)$,

$$V_{ij} = 2\frac{\partial^2(1/H)}{\partial x_i \partial x_j}H, \tag{5}$$

$$W = \frac{4}{\sqrt{H}}\sum_n e_n \frac{\partial^2 H}{\partial x_n^2}. \tag{6}$$

The contraction of equations (4) gives

$$\frac{6}{H^3}\sum_n e_n \frac{\partial^2 H}{\partial x_n^2} = \varkappa\rho. \tag{7}$$

The necessary and sufficient compatibility condition for the overdetermined system (4) is that the conformal function H satisfy the following system of 6 equations, containing only the function H and its derivatives:

$$V_{ij} = (e_{iW} + V_{ii})(e_{jW} + V_{jj}), \quad i \neq j. \tag{8}$$

In fact, if the function H satisfies Einstein' s equations (4), then from the equations in which $i = j$ we obtain

$$u_i = \sqrt{(e_i W + V_{ii})/\varkappa\rho} \tag{9}$$

and, substituting (9) into the equations with indices $i \neq j$, we arrive at equalities (8). If the function H is a solution of the system (8), then, conversely, by specifying u_i according to formulas (9), we obtain the energy-momentum tensor of that distribution of matter which produces the gravitational field with metric (3).

Consequently, the problem of conformally flat solutions of Einstein' s equations reduces to consideration of equations (7), (8), and (9). If the function H satisfies the system (8) and determines, according to (7), a positive function ρ, and according to (9) the real components of the 4-velocity u_i, then such a solution has physical meaning. We note that the vector specified by the components (9) in conformally flat space is always a unit vector $(u_i u^i = 1)$.

3. Let us consider the case when the conformal function H does not depend, at least, on one coordinate x_a. Then

$$V_{ia} = 0,$$

and in the system (8) there are equations

$$e_a W(e_i W + V_{ii}) = 0 \quad (i \neq a),$$

whose fulfillment is possible only in the case

$$W = 0, \tag{10}$$

since the case $(e_i W + V_{ii}) = 0$ leads to a contradiction with the remaining equations of the system (8). The system (8), equalities (7) and (9), under condition (10), take the form, respectively,

$$V_{ij}^2 = V_{ii} V_{jj}, \tag{8'}$$

$$\varkappa\rho = \frac{3}{H^4} \sum_n e_n \left(\frac{\partial H}{\partial x_n} \right)^2, \tag{7'}$$

$$u_i = \sqrt{V_{ii}/\varkappa\rho}. \tag{9'}$$

From equality $(7')$ and from the requirements of the material character of the function H and the positivity of the density ρ, it follows that among the coordinates on which the conformal function H depends there must necessarily be the time coordinate x_0.

4. Let us apply the formalism constructed to find the metric of Friedmann-Lobachevsky space. In this case, as the additional requirement, (2) is taken:

$$H = H(s), \quad s = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{11}$$

Substituting (11) into (5), we obtain

$$V_{ij} = 4H \left[e_i \delta_{ij} (1/H)' + 2 e_i e_j x_i x_j (1/H)'' \right], \tag{12}$$

where the prime denotes differentiation with respect to the parameter s.

Substituting (12) into (8), we obtain

$$W^2 + 8WH(1/H)' + 16H^2(1/H)'^2 =$$

$$= -8H(1/H)'' \left[W + 4H(1/H)' \right] (e_i x_i^2 + e_j x_j^2). \tag{13}$$

Since the left-hand side of (13) depends only on the parameter s, whereas the right-hand side contains coordinates not combined into such a group, the equality must be zero. Verification shows that the right-hand and left-hand sides of (13) simultaneously vanish only under the condition

$$W + 4H(1/H)' = 0, \tag{14}$$

which, in more detail, is written as follows:

$$2H\left(H'' + \frac{3}{2}\frac{1}{s}H'\right) - H'^2 = 0. \tag{14'}$$

The solution of equation (14) is the function

$$H = (c_2 + c_1/\sqrt{s})^2. \tag{15}$$

The choice of the integration constants c_1 and c_2 in accordance with the requirement $\rho > 0$ leads to metric (1).

What has been set forth shows that, under condition (11), metric (1) is the only conformally flat solution of Einstein' s equations with the energy-momentum tensor of dustlike matter. This removes the unfounded criticism of Fock' s solution in the work of Ya. I. Pugachev ([4]).

5. We shall give other conformally flat solutions of Einstein' s equations, obtained by the method described above and satisfying, for a suitable choice of arbitrary constants, the condition $\rho > 0$.

a) Let

$$H = H(z), \qquad z = k_0x_0 + k_1x_1 + k_2x_2 + k_3x_3, \tag{16}$$

where the k_i are arbitrary constants. Then equations (8′) are satisfied identically, while equation (10) gives

$$ds^2 = (k_0x_0 + k_1x_1 + k_2x_2 + k_3x_3 + c)^4(dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2). \tag{17}$$

In the space with metric (17), the density of the dust is

$$\varkappa\rho = \frac{12}{H^3}(k_0^2 - k_1^2 - k_2^2 - k_3^2), \tag{18}$$

the velocity of the dust particles is

$$u_i = \frac{k_i}{\sqrt{k_0^2 - k_1^2 - k_2^2 - k_3^2}}H. \tag{19}$$

b) Let the conformal function depend on one coordinate (therefore, on the time coordinate x_0). Then only one solution is possible, which is a special case of (17):

$$ds^2 = (k_0 x_0 + c)^4 (dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2). \qquad (17')$$

c) Let the conformal function depend on two coordinates. Then two types of solutions are possible: a special case of metric (17)

$$ds^2 = (k_0 x_0 + k_\alpha x_\alpha + c)^4 (dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2), \qquad (17'')$$

where $x_\alpha = x_1$, or x_2, or x_3, and a metric with plane symmetry in Taub' s sense ([5]) (the plane of symmetry is the plane $x_\alpha = \text{const}$)

$$ds^2 = \left[A + \frac{1}{\sqrt{B_1(x_0 - x_\alpha) + B_2}} + \frac{1}{\sqrt{C_1(x_0 + x_\alpha) + C_2}}\right]^4 (dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2), \qquad (20)$$

where A, B_1, B_2, C_1, C_2 are arbitrary constants, $\alpha = 1$, or 2, or 3.

d) Let

$$H = \frac{1}{\sum_n c_{nx} n} \varphi \left(\frac{\sum_n a_{nx} n}{\sum_n c_{nx} n} \right), \qquad (21)$$

where the a_i, c_i are arbitrary constants, and φ is an arbitrary function of the indicated argument. Then equations ($8'$) are satisfied identically. Consequently, the conformal function (21) will be a solution of Einstein' s equations if the function φ is chosen so that condition (10) is satisfied. Substitution of (21) into (10) leads to an ordinary differential equa-

equation

$$(\sqrt{\varphi})'' \cdot \sum_n e_n (c_n y - a_n)^2 + 3(\sqrt{\varphi})' \cdot \sum_n e_n (c_n^2 y - a_n c_n) + \frac{3}{4} \sqrt{\varphi} \cdot \sum_n e_n c_n^2 = 0, \quad (22)$$

where

$$y = \sum_n a_n x_n \Big/ \sum_n c_n x_n,$$

and the prime denotes differentiation with respect to y.

Among the solutions of equation (22), let us note the solution corresponding to the choice of constants $a_0 = -a_2 = c_1 = 1,\ a_1 = a_3 = c_0 = c_2 = c_3 = 0$, i.e.

$$y = \frac{x_0 - x_2}{x_1}, \qquad \varphi = \left(\frac{c_1}{\sqrt{y}} + \frac{c_2}{\sqrt{y^3}} \right)^2,$$

where c_1 and c_2 are constants of integration. In this case the metric has the form

$$ds^2 = \frac{[c_1(x_0 - x_2) + c_2 x_1]^4}{(x_0 - x_2)^6} \left(dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \right) \tag{23}$$

and admits a group of motions G_6, just as does metric (1) ([6], p. 318).

6. The examples considered, of course, do not exhaust the whole variety of conformally flat gravitational fields created by dust-like matter. We could assert that we know all solutions of this kind if the general solution of system (8) were known. It is clear, however, that conformally flat gravitational fields are numerous.

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Note: Figure translations are in progress. See original paper for figures.