SEMI-INTEGRAL AND QUASI-NUCLEAR MAPPINGS

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SEMI-INTEGRAL AND QUASI-NUCLEAR MAPPINGS

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In the present note a connection is established between quasi-nuclear ([4]) and semi-integral (([2]), Ch. I, Definition 8) mappings. In addition, along with quasi-nuclear mappings, we study mappings that are p-quasi-nuclear for $0 < p \leq 2/3$ and q-quasi-nuclear on the left and on the right for $2/3 < q \leq 1$.

I. Definition 1 (([5]), 2.1.1). Let E, F be locally convex spaces. A continuous linear mapping $A : E \to F$ is called **absolutely summing** if, for every summable sequence (x_i) in E, the sequence (Ax_i) is absolutely summable in F.

As Pietsch showed (([5]), 2.2.1), in the case when E, F are Banach spaces, a mapping $A : E \to F$ is absolutely summing if and only if there exists $\rho \geq 0$ such that

$$\sum_{i=1}^{n} \|Ax_i\| \leq \rho \sup_{\|x'\| \leq 1} \sum_{i=1}^{n} |\langle x_i, x' \rangle|$$

for arbitrary finite sets $x_1, \dots, x_n \in E$. The absolutely summing norm of A is denoted by $\pi(A) = \inf \rho$ over all such ρ (see ([5]), 2.2.2).

If E, F are locally convex spaces, then, as is known, $E \otimes F$ is a vector subspace of the space $L(E'_\tau, F)$ of all continuous linear mappings from E'_τ into F, where E'_τ is the dual of E with the Mackey topology. Consider on $L(E'_\tau, F)$ the topology ε of uniform convergence on equicontinuous sets in E'. The topology on $E \otimes F$ induced from $L_\varepsilon(E'_\tau, F)$ is called the ε-topology; $E \otimes F$ with this topology is denoted by $E \otimes_\varepsilon F$, and its completion by $E \hat{\otimes}_\varepsilon F$. We shall denote the projective tensor product of the spaces E and F (([2]), Ch. I, Definition 2) by $E \otimes_\pi F$, and its completion by $E \hat{\otimes}_\pi F$. Modifying Grothendieck' s arguments on the representation of $l^1 \hat{\otimes} E$ (([2]), Ch. I, § 3) and $l^1 \hat{\otimes} F$ (([2]), Ch. I, Theorem 2), we obtain the following

Proposition 1. Let E, F be locally convex spaces and let $A : E \to F$ be linear. If F is complete and $1 \otimes A : l^1 \otimes_\varepsilon E \to l^1 \otimes_\pi F$ is continuous, then A is absolutely summing. If E and F are Fréchet spaces and $A : E \to F$ is absolutely summing, then $1 \otimes A : l^1 \otimes_\varepsilon E \to l^1 \otimes_\pi F$ is continuous. Moreover, if E and F are Banach spaces, $\pi(A) = \|1 \otimes A\|$.

With the aid of Proposition 1 the following is established.

Theorem 1. Let E, F be Fréchet spaces and let $A : E \to F$ be linear. In order that A be absolutely summing, it is necessary and sufficient that, for every compact mapping $u : c_0 \to E$, the composition $A \circ u$ be nuclear. In the case when E and F are Banach spaces,

$$\pi(A) = \sup_{\|u\| \le 1} \|A \circ u\|_1,$$

where $\|f\|_1$ denotes the trace norm of the nuclear mapping f (([2]), Ch. I, Definition 4).

II. Let E, F be Banach spaces and $0 < s \le 1$. For every $u \in E \otimes F$ put

$$\|u\|_s = \inf \left(\sum |\lambda_i|^s \right)^{1/s},$$

where the infimum is taken over all representations of u in the form $\sum \lambda_i x_i \otimes y_i$ with $\|x_i\| \le 1$, $\|y_i\| \le 1$. The function $u \mapsto \|u\|_s$ defines on $E \otimes F$ a metric topology agreeing with the structure

of the vector space into $E \otimes F$. The completions of $E \otimes F$ in this topology are denoted by $E \overset{(s)}{\otimes} F$ (see ([2]), Ch. 2, §1). Let now $B(E, F)$ be the space of all continuous bilinear forms on $E \times F$, and

$$\omega_s : E' \overset{(s)}{\otimes} F' \to B(E, F)$$

the canonical mapping. Bilinear forms contained in $\omega_s(E' \overset{(s)}{\otimes} F')$ will be called s-nuclear. For each

$$\tilde{u} \in \omega_s(E' \overset{(s)}{\otimes} F')$$

we shall call the number

$$\|\tilde{u}\|_s = \inf_{v \in \omega_s^{-1}(u)} \|u + v\|_s,$$

where $\omega_s(u) = \tilde{u}$, the s-nuclear norm* of $\tilde{u}$.

Definition 2. Let E, F be Banach spaces, $0 < p \le 2/3$ and $2/3 < q \le 1$. A bilinear form u on $E \times F$ will be called p-nuclear at a glance (q-quasinuclear on the left (on the right)) if the bilinear form

$$l^1(I) \times l^1(J) \xrightarrow{\varphi \times \psi} E \times F \xrightarrow{u} R \quad \left(l^1(I) \times F \xrightarrow{\varphi \times 1} E \times F \xrightarrow{u} R \quad \left(E \times l^1(J) \xrightarrow{1 \times \psi} E \times F \xrightarrow{u} R \right) \right)$$

is p-nuclear (q-nuclear) for some normed homomorphisms "onto"

$$\varphi : l^1(I) \to E \quad \text{and} \quad \psi : l^1(J) \to F;$$

the p-quasinuclear (q-quasinuclear on the left (on the right)) norm of u (notation $\|u\|_{p1}(\|u\|_{q0}(\|u\|_{0q})))$ is defined to be the p-nuclear (q-nuclear) norm of the bilinear form

$$u \circ (\varphi \times \psi) \quad (u \circ (\varphi \times 1)(u \circ (1 \times \psi)))\,.$$

A mapping $\tilde{u} : E \to F$ will be called p-quasinuclear (q-quasinuclear on the left (on the right)) if the corresponding bilinear form on $E \times F'$ is p-quasinuclear (q-quasinuclear on the left (on the right)); the p-quasinuclear norm (q-quasinuclear on the left (on the right)) of the mapping $\tilde{u}$ will be called the p-quasinuclear (q-quasinuclear on the left (on the right)) norm of the corresponding bilinear form on $E \times F'$ (notation

$$\|\tilde{u}\|_{p1}(\|\tilde{u}\|_{q0}(\|\tilde{u}\|_{0q})).$$

1-quasinuclear mappings on the left (on the right) will be called quasinuclear on the left (on the right). It is not difficult to verify that quasinuclear mappings on the right coincide with the quasinuclear mappings in the sense of Pietsch ([4]), and their quasinuclear-on-the-right and quasinuclear (([5]), 3.2.3) norms are equal. By analogy with the definition of p-quasinuclear mappings for $0 < p \leq 2/3$, one can give a definition of q-quasinuclear mappings for $2/3 < q \leq 1$. However, this goes beyond the class of mappings whose composition in any finite number are nuclear. Moreover, it can be shown that every compact mapping

$$u : l^2 \to l^2$$

has the property that, for arbitrary normed homomorphisms

$$\varphi : l^1(I) \to l^2 \quad \text{and} \quad \psi : l^1(J) \to l^2,$$

the composition

$$l^1(I) \xrightarrow{\varphi} l^2 \xrightarrow{u} l^2 \xrightarrow{{}^t\psi} l^\infty(J)$$

is nuclear.

In view of this, in what follows we shall assume that $0 < p \leq 2/3$ and $2/3 < q \leq 1$. We indicate the basic properties of p-quasinuclear and q-quasinuclear on the left (on the right) mappings.

Proposition 2. If E, F, G, H are Banach spaces, $u : F \to G$ is p-quasinuclear (q-quasinuclear on the left (on the right)) and $\alpha : E \to F$, $\beta : G \to H$ are continuous, then $\beta u \alpha$ is p-quasinuclear (q-quasinuclear on the left (on the right)), and

$$\|\beta u\alpha\|_{p1} \leq \|\beta\|\,\|u\|_{p1}\|\alpha\| \quad \left(\|\beta u\alpha\|_{q0} \leq \|\beta\|\,\|u\|_{q0}\|\alpha\| \quad \left(\|\beta u\alpha\|_{0q} \leq \|\beta\|\,\|u\|_{0q}\|\alpha\|\right)\right).$$

Proposition 3. Let E, F, G, H be Banach spaces, let $\alpha : E \to F$ be a normed homomorphism "onto," $\beta : G \to H$ an isometry "into," and let $u : F \to G$ be such that

$$E \overset{\alpha}{\to} F \overset{u}{\to} G \overset{\beta}{\to} H$$

is p-quasinuclear

$$\left(E \overset{\alpha}{\to} F \overset{u}{\to} G \text{ is } q\text{-quasinuclear on the left } \left(F \overset{u}{\to} G \overset{\beta}{\to} H \text{ is } q\text{-quasinuclear on the right} \right) \right).$$

Then u is p-quasinuclear (q-quasinuclear on the left (on the right)), and

$$\|u\|_{p1} = \|\beta u \alpha\|_{p1} \quad \left(\|u\|_{q0} = \|u\alpha\|_{q0} \quad \left(\|u\|_{0q} = \|\beta u\|_{0q} \right) \right).$$

Theorem 2. Let E, F be Banach. A continuous linear mapping

$$u : E \to F$$

is p-quasinuclear (q-quasinuclear on the left (on the right)) if and only if

$${}^t u : F' \to E'$$

is p-quasinuclear (q-quasinuclear on the right (on the left)), and

$$\|u\|_{p1} = \|{}^t u\|_{p1} \quad \left(\|u\|_{q0} = \|{}^t u\|_{0q} \quad \left(\|u\|_{0q} = \|{}^t u\|_{q0} \right) \right).$$

* For $0 < s < 1$ this is not a norm in the usual sense, since $\|\tilde{u}+\tilde{v}\|_s \leq C_s(\|\tilde{u}\|_s+\|\tilde{v}\|_s)$,

where C_s is a constant depending on s.

Let E, F be Banach spaces. Denote by $N_{p1}(E,F)$ ($N_{q0}(E,F)$ ($N_{0q}(E,F)$)) the spaces of all p-quasi-nuclear (left (right) q-quasi-nuclear) mappings with the topology defined by the p-quasi-nuclear (left (right) q-quasi-nuclear) norm.

Proposition 4. *The spaces $N_{p1}(E,F)$, $N_{q0}(E,F)$, $N_{0q}(E,F)$ are complete metric spaces, and if E' or F satisfies the approximation condition* $((^2), Ch.I, Definition 9)$, *then $E' \otimes F$ is dense in $N_{p1}(E,F)$, $N_{10}(E,F)$, $N_{0q}(E,F)$.*

Corollary. *Every right quasi-nuclear mapping into the space $C(K)$ of all continuous functions on a compactum K is nuclear, and the quasi-nuclear and nuclear norms of this mapping coincide.*

III. **Definition 3** $((^2), Ch.I, Definition 8)$. Let E, F be Banach spaces. A mapping $A : E \to F$ is called **left (right) semi-integral** if the mapping $1 \otimes A : c_0 \otimes_\varepsilon E \to c_0 \otimes_\pi F$ ($1 \otimes A : l^1 \otimes_\varepsilon E \to l^1 \otimes_\pi F$) is continuous. The **left (right) semi-integral norm** of A is the norm of the continuous mapping $1 \otimes A$ (notation $\|A\|'_{10}$ ($\|A\|'_{01}$)).

It follows from Proposition 1 that, if E and F are Banach spaces, then $A : E \to F$ is absolutely summing if and only if A is right semi-integral, and moreover $\|A\|'_{01} = \pi(A)$. Since l^∞ satisfies the metric approximation condition $((^2), Ch.I, Proposition 41)$, it follows that $\|A\|'_{10} = \|A\|_{10}$ ($\|A\|'_{01} = \|A\|_{01}$). Thus, for every right quasi-nuclear mapping A we obtain $\|A\|_{01} = \pi(A)$ (see also ([4])).

Let us proceed to establish the connection between quasi-nuclear and semi-integral mappings.

Theorem 3. *Let E, F be Banach spaces and $A : E \to F$ a linear mapping. If A is right semi-integral, then for every weakly compact ball K in E the composition*

$$E_K \xrightarrow{k} E \xrightarrow{A} F$$

is right quasi-nuclear and

$$\|A \circ u_k\|_{01} \le \|A\|'_{01}\|u_k\|,$$

where u_k is canonical. If for every compact ball K in E the composition

$$E_K \xrightarrow{k} E \xrightarrow{A} F$$

is right quasi-nuclear, then A is right semi-integral and

$$\|A\|'_{01} \le \sup_{\|u_k\| \le 1} \|A \circ u_k\|_{01}.$$

Corollary 1. *Every right semi-integral mapping from a reflexive space into an arbitrary Banach space is right quasi-nuclear.*

Corollary 2. *Let E, F be Banach spaces and $A : E \to F$ linear. If there exist a $\sigma(E', E'')$-compact ball M in E' and a positive measure μ on M such that*

$$\|Ax\| \le \int_M |\langle x, x'\rangle|\, d\mu(x')$$

for every $x \in E$, then A is right quasi-nuclear.

Theorem 4. *Let E, F be Banach spaces. A mapping $A : E \to F$ is left semi-integral if and only if for every compact ball K in E the composition*

$$E_K \xrightarrow{u_k} E \xrightarrow{A} F$$

is left quasi-nuclear, and

$$\|A\|'_{10} = \sup_{\|u_k\| \le 1} \|A \circ u_k\|_{10},$$

where u_k is canonical.

Theorem 5. *Let E, F be Banach spaces. In order that $A : E \to F$ be right semi-integral, it is necessary and sufficient that for every compact mapping $u : F \to c_0$ the composition $u \circ A$ be nuclear; moreover*

$$\|A\|'_{01} = \sup_{\|u\| \leq 1} \|u \circ A\|_1.$$

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Note: Figure translations are in progress. See original paper for figures.