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HYDROMECHANICS

1969

SovietRxiv

Abstract

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UDC 533.601.135+517.934

HYDROMECHANICS

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ON THE THEORY OF UNSTEADY TRANSONIC FLOWS

(Presented by Academician M. A. Lavrent' ev on 16 VII 1968)

It is known that the system of equations of hydrodynamics describing the motion of a gas can, under the assumption that the flow is potential, be replaced by a partial differential equation for the potential. In considering transonic flows, the original equation can be approximately replaced by the quasilinear equation

$$-\varphi_x\varphi_{xx} + \varphi_{yy} - 2\varphi_{xt} = 0. \tag{1}$$

Here all variables—the unknown function φ, the time t, and the spatial variables x, y—are dimensionless. Subscripts denote partial derivatives. The nonlinear equation (1) was first obtained in (^{1}), where oscillations of a thin airfoil in a transonic flow were considered. Equation (1) is hyperbolic at each point x, y, t. However, whereas for the exact equation the fundamental problem is the Cauchy problem—in which at the initial instant the potential and its time derivative are prescribed—for equation (1) this problem is incorrect, since the planes $t = \text{const}$ are characteristics. In the present paper a boundary-value problem for equation (1) is formulated, and uniqueness of its solution in a certain class of functions is proved. A linear model of the equation under consideration is also proposed. It is proved that the boundary-value problem posed for the model is well posed.

We shall restrict ourselves to consideration of the domain $t \geq 0,\ x \geq 0,\ -\infty < y < \infty$. For equation (1) we pose the following problem: in the indicated domain it is required to find a solution $\varphi(x, y, t)$ of this equation satisfying the conditions

$$\varphi(x, y, 0) = \varphi_0(x, y), \qquad \varphi(0, y, t) = \psi_0(y, t), \qquad \frac{\partial\varphi}{\partial x}(0, y, t) = \psi_1(y, t). \tag{2}$$

Here $\varphi_0(x, y)$, $\psi_0(y, t)$, $\psi_1(y, t)$ are prescribed sufficiently smooth functions. In addition, we require that the inequality $\psi_1(y, t) > 0$ hold for $t \geq 0,\ -\infty < y < \infty$. This inequality means that the plane $x = 0$ is a manifold of space-like type for equation (1).

We shall prove that the problem (2) posed has a unique solution in the class of functions satisfying, in the domain under consideration, the condition $|\varphi_x| \leq M$ with arbitrary constant M.

If φ_1 and φ_2 are two solutions of the problem, then their difference $u = \varphi_1 - \varphi_2$ satisfies the linear equation

$$-\varphi_{1x}u_{xx} + u_{yy} - 2u_{xt} - \varphi_{2xx}u_x = 0 \tag{3}$$

and homogeneous boundary conditions. Denote $-\varphi_{1x} = f$, $-\varphi_{2xx} = l$, and instead of (3) consider the equation

$$fu_{xx} + u_{yy} - 2u_{xt} + lu_x = 0. \tag{4}$$

We note that $|f(x, y, t)| \leq M$.

The last inequality makes it possible to single out a family of planes of space-like type with normal components $M, 1, 0$ along the axes t, x, y, respectively. The point is that, although the plane $x = 0$, carrying the initial

data—of spatial type; when the coefficient f vanishes and changes sign, the planes $x = \text{const}$ cease to be so, and we cannot use the standard technique of a priori estimates for hyperbolic equations. The existence of the above-mentioned family nevertheless makes it possible to resort to this technique. The fact that the planes with normal $\xi = (M, 1, 0)$ do indeed form a family of planes of spatial type is easily verified by means of the definition formulated in (∗).

It is convenient to rotate the axes t, x so that the indicated family of planes, in the new variables $\bar{t}, \bar{x}, y$, becomes the family $\bar{t} = \text{const}$. Then equation (4) is transformed into the form

$$u_{tt} - \frac{fM^2 + 2M}{2M - f}u_{xx} + 2\frac{M^2 - fM - 1}{2M - f}u_{xt} + \frac{M^2 + 1}{2M - f}u_{yy} + ... = 0. \tag{5}$$

Here the lower-order terms have not been written out explicitly, and the bars over the independent variables and the function f have been omitted. After the rotation, the original domain passes into the interior of the dihedral angle enclosed between the planes $t + Mx = 0$ and $x - Mt = 0$ (with $t \geq 0$).

Consider the domain Ω obtained as the intersection of this dihedral angle and another dihedral angle. The edge of the latter is parallel to the x-axis and lies in the plane $t = \text{const} > 0$, while its faces are directed toward negative values of t. In this domain, by the standard method, we prove uniqueness of the solution of the problem; i.e., we multiply equation (5) by the "separating" first-order operator $u_t + a_{10}u_x$ (a_{10} denotes one half of the coefficient of the mixed derivative), integrate the resulting equality over the domain Ω, "truncated" by the plane $t = \text{const}$, use the Ostrogradsky-Gauss formula, and so on. In doing this, the inclination of the faces of the latter dihedral angle is chosen so that the surface integrals over them are nonnegative.

If, in problem (2), walls $y = C_1$ and $y = C_2$ are added (C_1 and C_2 being certain constants), then the proof of uniqueness is carried out in a completely analogous way. On the walls one may prescribe the values of the solution φ itself or of its normal derivative.

We note that these arguments do not depend on the number of spatial variables and therefore also apply to the case of three variables x, y, z.

Passing through the speed of sound corresponds to a change of sign of the derivative φ_x; therefore we shall be interested in the case where this derivative vanishes on some regular surface in the space x, y, t. As one of the simplest models of equations (1) possessing this property, we shall consider the equation

$$xu_{xx} + u_{yy} - 2u_{xt} = 0. \tag{6}$$

The analogue of problem (2) for the linear model (6) consists in prescribing the solution u for $t = 0$, and also prescribing the solution itself and its derivative with respect to x on the plane $x = C = \text{const}$, with $C < 0$, for example on the plane $x = -1$. The solution is sought in the domain $\bar{D}$: $t \geq 0,\ x \geq -1,\ -\infty < y < \infty$.

Introducing the new variables

$$v = ue^{\frac{1}{2}t}; \qquad x_1 = e^{-\frac{1}{2}t} + xe^{\frac{1}{2}t}; \qquad t_1 = e^{-\frac{1}{2}t} - xe^{\frac{1}{2}t}; \qquad y_1 = y$$

transforms equation (6) into the equation

$$v_{t_1t_1} - v_{x_1x_1} - v_{y_1y_1} = 0. \tag{7}$$

The domain $\bar{D}$ passes into a new domain D_1 such that the characteristic cone of equation (7) cuts out finite domains on the surfaces carrying the initial data. Hence it follows that the mixed boundary-value problem posed for equation (6) is well posed.

Following (3), one can compute the operators of the broadest group of transformations admitted by equation (1). The basis of the corresponding Lie algebra has the form

$$\begin{aligned}
X_1 &= y\,\partial/\partial y + 2t\,\partial/\partial t - 2\varphi\,\partial/\partial\varphi,\\
X_2 &= [\alpha'(t)x + \alpha''(t)y^2]\,\partial/\partial x + 2\alpha'(t)y\,\partial/\partial y + 3\alpha(t)\,\partial/\partial t +\\
&\quad + [-\alpha'(t)\varphi + \alpha''(t)x^2 + 2\alpha'''(t)xy^2 + \tfrac{1}{3}\alpha^{(4)}y^4]\,\partial/\partial\varphi,\\
X_3 &= \beta'(t)y\,\partial/\partial x + \beta(t)\,\partial/\partial y + [2\beta''(t)xy + \tfrac{2}{3}\beta'''(t)y^3]\,\partial/\partial\varphi,\\
X_4 &= \gamma(t)\,\partial/\partial x + [2\gamma_1(t)x + 2\gamma''(t)y^2]\,\partial/\partial\varphi,\\
X_5 &= m(t)y\,\partial/\partial\varphi,\\
X_6 &= n(t)\,\partial/\partial\varphi,
\end{aligned}$$

i.e., its components depend on 5 arbitrary functions of time $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $m(t)$, $n(t)$. Primes denote differentiation.

The obtained groups of transformations admitted by equation (1) make it possible to construct a number of its particular solutions. The presence of the operators X_5 and X_6, for example, means that the equation preserves its form when any function of the form $m(t)y + n(t)$ is added to the solution.

In the case of three spatial variables, some particular solutions of this equation have been found from other considerations in (4).

The author expresses his gratitude to L. V. Ovsyannikov for his constant attention to the work.

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Received
9 VII 1968

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