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MATHEMATICS

1969

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Abstract

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UDC 62.501 + 517.941

MATHEMATICS

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ON A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS FOR WHICH QUESTIONS OF STABILITY IN THE LARGE AND OF INSTABILITY CAN BE SOLVED EFFECTIVELY

(Presented by Academician V. I. Smirnov on 19 IX 1968)

1°. Consider a system, which we shall denote by (S, φ) , consisting of a linear part with equations

$$dx/dt = Px + q\xi, \quad \sigma = r^*x, \quad (S)$$

and a nonlinear part with equation $\xi = \varphi[t, \sigma(\tau)|_{\tau=0}^t]$. Here P, q, r are constant real matrices of orders, respectively, $\nu \times \nu, \nu \times n, \nu \times m$; x is a solution vector of order ν ; the vectors ξ, φ, σ have orders n, n , and m . It is assumed that φ depends on t and $\sigma(\tau), 0 \leq \tau \leq t$, in such a way that:

(α) for any $x(0)$ there exists an (absolutely continuous) solution $x(t), 0 \leq t < \infty$.

It is also possible that φ is a hysteresis function of σ in the sense of definition 1⁽¹⁾ with properties 1°, 2° (1). It is assumed that:

(β) the matrix P has no eigenvalues on the imaginary axis;

(γ) the linear part (S) is completely controllable in the sense of Kalman, i.e. the rank of the $\nu \times m\nu$ matrix $R = \|q, Pq, \dots, P^{\nu-1}q\|$ is equal to ν^* .

Denote by $\chi(\lambda) = r^*(P - \lambda I)^{-1}q$ the transfer $m \times n$ -matrix from the inputs ξ to the outputs $(-\sigma)$. Let $F(\xi, \sigma, \dot{\sigma})$ be a real quadratic form of the vector arguments $\xi, \sigma, \dot{\sigma}$ of orders n, m, m . We shall say that the condition $\{S, \varphi, F \geq 0\}$ is satisfied if for every solution of the system (S, φ) and for the corresponding $\xi = \xi(t), \sigma = \sigma(t)$, one has $F(\xi, \sigma, d\sigma/dt) \geq 0$ for $t \geq 0$. We note that usually (2) the form F , for which $\{S, \varphi, F \geq 0\}$ is satisfied, is determined by the standard properties of the nonlinearities φ . Extend the form $F(\xi, \sigma, \dot{\sigma})$, preserving Hermiticity, to complex values of the arguments and put $F(\lambda, \bar{\varphi}) = F(\bar{\varphi}, \tilde{\sigma}, \lambda\tilde{\sigma})$, where $\tilde{\sigma} = -\chi(\lambda)\bar{\varphi}$, $\bar{\varphi}$ is a complex vector. We shall say that the condition $\{F\}$ is satisfied if $F(i\omega, \bar{\varphi}) < 0$ for all $\bar{\varphi} \neq 0$ and all $-\infty \leq \omega \leq +\infty^{**}$.

Consider, for a fixed form F , the class of systems (S, φ) satisfying the conditions $\{F\}$ and $\{S, \varphi, F \geq 0\}$. It turns out that all systems of this class are simultaneously either exponentially stable in the large, or exponentially unstable. Therefore, if from a system (S, φ) of this class it is possible, by changing the nonlinear part, to obtain an integrable system (S, ψ) (for example, with a linear function ψ) of the same class, then the question of stability (instability) of the original system is solved. We pass to precise formulations.

* All the results given below carry over to the case when either (β) or (γ) is not satisfied, but the formulations of the results become more cumbersome. Unless this is specially stated, we shall assume that for all systems of the form (S, φ) considered below, $(\alpha), (\beta), (\gamma)$ are satisfied.

** Diverse examples of systems (S, φ) for which the conditions $\{S, \varphi, F \geq 0\}, \{F\}$ are satisfied are given in ⁽²⁾. (In contrast to ⁽²⁾, it may be assumed that in these examples the poles of $\chi(\lambda)$ need not lie in the left half-plane.)

Let $C > 0, \varepsilon > 0$ be certain numbers. Consider the following qualitative types of behavior of solutions:

$(A_{C,\varepsilon})$ for any solution $x(t)$ of the system (S, φ) and any $t \geq t_0 \geq 0$ one has

$$|x(t)| \leq C e^{-\varepsilon(t-t_0)} |x(t_0)|$$

and $\zeta(t) \in L_2(0, \infty)$;

$(B_{C,\varepsilon})$ in the space $\{x\}$ there exists a cone K of the form $x^* H_0 x < 0$ (where $H_0 = H_0^*$) such that for any solution $x(t)$ of the system (S, φ) satisfying the condition $x(0) \in K$, one has

$$|x(t)| \geq C e^{\varepsilon(t-t_0)} |x(t_0)|$$

for any $t \geq t_0 \geq 0$.

Theorem 1. *Suppose that for the system (S, φ) the conditions $\{F\}, \{S, \varphi, F \geq 0\}$ are satisfied. Then only the two above-indicated types of behavior of solutions of the system (S, φ) are possible: either $(A_{C,\varepsilon})$, or $(B_{C,\varepsilon})$. Here the numbers $C > 0, \varepsilon > 0$ depend only on the coefficients of the linear part (S) and on the form F .*

Theorem 2. *Consider two systems (S, φ) and (S, ψ) , differing only in their nonlinear parts. Suppose that the conditions $\{F\}, \{S, \varphi, F \geq 0\}, \{S, \psi, F \geq 0\}$ are satisfied. Then both these systems have the same type of behavior of solutions: either $(A_{C,\varepsilon})$ holds for both systems, or $(B_{C,\varepsilon})$ holds for both systems.*

A number of results follow from Theorems 1 and 2 ⁽²⁾.

The following theorem shows that under a continuous deformation of the linear part (S) and of the form F , with the conditions $\{F\}, \{S, \varphi, F \geq 0\}$ preserved, each of the two indicated types of behavior of solutions does not change. This

makes it possible to determine the type of behavior of solutions of the original system by reducing the original system to an integrable system not only by changing its nonlinear part, but also by changing its coefficients. Namely, consider the system (S, φ) with coefficients $P = P_\xi$, $q = q_\xi$, $r = r_\xi$ and with nonlinear part

$$\zeta = \varphi_\xi[t, \sigma(\tau)|_{\tau=0}^t],$$

depending on ξ , $0 \leq \xi \leq 1$. Denote this system by (S_ξ, φ_ξ) . Let the coefficients of the quadratic form F_ξ also depend on ξ . Construct the form

$$F_\xi(\lambda, \tilde{\zeta}) = F_\xi(\tilde{\zeta}, \sigma, \lambda \tilde{\sigma}),$$

where

$$\tilde{\sigma} = \chi_\xi(\lambda), \quad \chi_\xi(\lambda) = r_\xi^*(P_\xi - \lambda I)^{-1} q_\xi.$$

Theorem 3. *Suppose that the coefficients of the form F_ξ , and also P_ξ , q_ξ , r_ξ , depend continuously on ξ , $0 \leq \xi \leq 1$. Let, for $0 \leq \xi \leq 1$, the conditions (F_ξ) , $(S_\xi, \varphi_\xi, F_\xi \geq 0)$ be satisfied. Then there exist numbers $C > 0$, $\varepsilon > 0$, independent of ξ , such that either for all systems (S_ξ, φ_ξ) , $0 \leq \xi \leq 1$, $(A_{C, \varepsilon})$ holds, or for all these systems $(B_{C, \varepsilon})$ holds.*

We emphasize that in Theorem 3 no continuity of φ_ξ with respect to ξ is required.

It is interesting to note that the Nyquist criterion follows from Theorem 3.* Thus, Theorem 3 may be regarded as an extension of the Nyquist criterion to the class, indicated above, of, generally speaking, nonlinear systems (S, φ) .

Remark. Theorems 1-3, applied to the system obtained from the original one by the change

$$x_1 = x e^{\alpha t}, \quad \varphi_1 = \varphi e^{\alpha t},$$

make it possible to estimate the order of exponential decay of all solutions or exponential growth of some solutions of the original system. In this case the conditions $\{S, \varphi, F \geq 0\}$, $\{F\}$ pass into the conditions

$$F(\zeta, \sigma, \sigma + \alpha \sigma) \geq 0$$

for $t \geq 0$, and

$$F(i\omega - \alpha, \tilde{\varphi}) < 0$$

for $\tilde{\varphi} \neq 0$, $-\infty \leq \omega \leq +\infty$.

2°. Consider, in the previous notation, the system

$$dx/dt = Px + q\varphi[t, \sigma(\tau)|_{\tau=0}^t] + f(t, x), \quad \sigma(t) = r^*x(t), \quad (D)$$

where the vector-function $f(t, x)$ of order v satisfies the conditions

$$\lim_{|x| \rightarrow 0} |f(t, x)|/|x| = 0$$

uniformly in $t \geq 0$.

* In this case $m = n = 1$, $\varphi = \mu\sigma$, $\mu = \text{const}$. As F one should take

$$F = -|\varphi - \mu\sigma|^2.$$

The condition $\{S, \varphi, F \geq 0\}$, obviously, is satisfied. Since

$$F = -|1 + \mu\chi(i\omega)|^2|\tilde{\varphi}|^2,$$

$\{F\}$ means that

$$\chi(i\omega) \neq -\mu^{-1}, \quad -\infty \leq \omega \leq +\infty.$$

Theorem 4. Suppose that the system (S, φ) , which is obtained from (D) when $f(t, x) \equiv 0$, satisfies the conditions $\{S, \varphi, F \geq 0\}$, $\{F\}$ for some form F .

I. Let $(A_{C, \varepsilon})$ hold for the system (D) when $f \equiv 0$. Then in the space $\{x\}$ there exists an ellipsoid $\mathfrak{E}\{x^*H_0x < \text{const}\}$, where $H_0 = H_0^* > 0$, such that for the solutions of the system (D) the following hold: a) from $x(t_0) \in \mathfrak{E}$ it follows that $x(t) \in \mathfrak{E}$ for $t \geq t_0$; b) for every solution $x(t)$ there is a $t_0 > 0$ for which $x(t_0) \in \mathfrak{E}$; c) if the system (D) is given for $-\infty < t < +\infty$ and the property $\{S, \varphi, F \geq 0\}$ is fulfilled for every t , then there exists a solution $x^0(t) \in \mathfrak{E}$ for $-\infty < t < +\infty$; d) if $\varphi[t, \sigma(\tau)|_{\tau=0}] = \varphi[t, \sigma(t)]$ is an ordinary function and $\varphi(t+T, \sigma) = \varphi(t, \sigma)$, $f(t+T, x) = f(t, T)$, then there exists a T -periodic solution $x_0(t)$ of the system (D) such that $x^0(t) \in \mathfrak{E}$.

II. Let $(B_{C, \varepsilon})$ hold for the system $\{D\}$ when $f \equiv 0$. Then in the space $\{x\}$ there exists a domain $\mathfrak{G}\{|x| \geq \text{const}, x^*H_0x > \text{const}\}$, where $H_0 = H_0^*$, such that, for $x(t_0) \in \mathfrak{G}$, the inequality $|x(t)| \geq \text{const} \cdot e^{\varepsilon t}$, $\varepsilon > 0$, holds.

3°. Consider again the system (S, φ) .

Lemma 1. Let $V(x)$ be a continuous function, $V(0) = 0$, and let the derivative dV/dt along the system (S, φ) satisfy the inequality $dV/dt \leq -W(x, \zeta)$, where $W(x, \zeta)$ is a positive definite quadratic form in x and ζ .

I. If $V[x(t)] \geq 0$ for some solution $x(t)$ and all $t \geq 0$, then $x(t) \rightarrow 0$ as $t \rightarrow \infty$, and $|x| \in L_2(0, \infty)$, $|\zeta| \in L_2(0, \infty)$.

II. If $x(t) \rightarrow 0$ as $t \rightarrow \infty$, then $V[x(t)] > 0$ for $x(t) \neq 0$, $t \geq 0$.

III. If $x(t) \nrightarrow 0$ as $t \rightarrow \infty$, then the function $|x(t)|$ is unbounded and $V[x(t)] \rightarrow -\infty$ as $t \rightarrow \infty$.

Proof. I. We have successively: $\dot{V} \leq 0$, $V[x(t)] \rightarrow V_\infty \neq \infty$, $(-V) \in L(0, \infty)$, and, since $(-V) \geq \text{const}(|x|^2 + |\zeta|^2)$, then $|x|, |\zeta| \in L_2(0, \infty)$. From (S) it follows that $|dx/dt| \in L_2(0, \infty)$, $|x^*dx/dt| \in L(0, \infty)$. Consequently, there exists a limit of $|x(t)|$ as $t \rightarrow \infty$, which can only be zero. Assertion II is obvious. III. If the function $V[x(t)]$ is bounded below, then, repeating the proof of I, we obtain that $|x(t)| \rightarrow 0$ as $t \rightarrow \infty$. Since $\dot{V} \leq 0$, it follows that $V[x(t)] \rightarrow -\infty$ as $t \rightarrow \infty$.

Lemma 2. Let P, q be given matrices of orders $\nu \times \nu$, $\nu \times n$; let $H = H^*$ be the sought $\nu \times \nu$ matrix;

$$W(z, \zeta) = -2 \operatorname{Re} z^* H (Pz + q\zeta) + U(z, \zeta),$$

where z, ζ are vectors of orders ν and n ; $U(z, \zeta)$ is a Hermitian form, and the rank of the $\nu \times n\nu$ matrix $\|q, Pq, \dots, P^{\nu-1}q\|$ is ν . Let $P_\omega = P - i\omega I$ and $\det P_\omega \neq 0$, $-\infty < \omega < +\infty$. For the existence of a matrix $H = H^*$ such that $W(z, \zeta)$ is a positive definite form in z, ζ , it is necessary and sufficient that, for all $-\infty \leq \omega \leq +\infty$, the form $U_\omega(\zeta) = U(P_\omega^{-1}q\zeta, \zeta)$ be positive definite.

Proof. Since $W(P_\omega^{-1}q\zeta, \zeta) = U_\omega(\zeta)$, the indicated condition is necessary. Sufficiency follows from result (3), according to which (after a change of notation) the condition $U_\omega(\zeta) \geq 0$, $\forall -\infty \leq \omega \leq +\infty$, is sufficient for $W(z, \zeta) > 0$. Therefore, for the fulfillment of $W(z, \zeta) - \varepsilon\{|z|^2 + |\zeta|^2\} \geq 0$ it is sufficient that

$$U_\omega(\zeta) - \varepsilon\{\zeta^* q_\omega^* q_\omega \zeta + |\zeta|^2\} \geq 0,$$

where $q_\omega = P_\omega^{-1}q$, $-\infty \leq \omega \leq +\infty$. From the assumption concerning $U_\omega(\zeta)$ it follows that the last inequality is indeed fulfilled for small $\varepsilon > 0$.

4° Proof of Theorems 1-3. Construct the function $V(x) = x^* H x$ satisfying Lemma 1. We have $\dot{V} = -W - F$, where $W = -2x^* H (Px + q\zeta) - F$. Using Lemma 2, we obtain that the condition for the existence of a matrix $H = H^*$ for which W is a positive definite form in x and ζ is the condition $\{F\}$, which is fulfilled. From $\{S, \varphi, F \geq 0\}$ it follows that $\dot{V} \leq -W$. Therefore there exists a $V(x) = x^* H x$ satisfying the conditions of Lemma 1. One of two cases is possible: (A) for every solution of the system $\{S, \varphi\}$ one has $x \rightarrow 0$ as $t \rightarrow \infty$; (B) there is a solution $x^0 \neq 0$

as $t \rightarrow \infty$. Suppose that (A) holds. By Lemma 1, II, $V(x) > 0$, $\forall x \neq 0$. Then $W(x, \xi) \geq 2\varepsilon V(x)$, $\dot{V} \leq -2\varepsilon V$, $|x(t)| \leq C e^{-\varepsilon(t-t_0)} |x(t_0)|$ for some $C > 0$, $\varepsilon > 0$. By Lemma 1, I, $|\xi| \in L_2(0, \infty)$. Thus $(A_{C, \varepsilon})$ is satisfied. Suppose that (B) holds. By Lemma 1, III, $V[x^0(t)] \rightarrow -\infty$ as $t \rightarrow +\infty$. Consequently, the set $V(x) < 0$ is nonempty, and hence also the set (the cone K) $x^* H_0 x \equiv V(x) + \delta |x|^2 < 0$ for some small $\delta > 0$. Suppose $x(0) \in K$. Then $V[x(0)] < 0$, and since $\dot{V} \leq -W \leq 0$, it follows that $V[x(t)] < 0$, $t \geq 0$. Let $V(x) = V_1(x) - V_2(x)$, where $V_j(x)$ are nonnegative forms. We have $W \geq 2\varepsilon V_2$ for some small ε . Then for $x = x(t)$ we obtain $(-\dot{V}) \geq W \geq 2\varepsilon V_2 \geq 2\varepsilon(-V)$. Since $V \leq 0$ for $t \geq 0$, it follows that $-V(x) \geq \exp[2\varepsilon(t-t_0)] \cdot \{-V[x(t_0)]\} \geq \delta \exp[2\varepsilon(t-t_0)] \cdot |x(t_0)|^2$ for $t \geq t_0 \geq 0$. Since $-V(x) \leq \text{const} \cdot |x|^2$, $(B_{C, \varepsilon})$ is satisfied. The matrix H , and hence the numbers C, ε , depend only on P, q, r and on the coefficients of the form F . Theorem 1 is proved. It follows from what has been set forth that the matrix H is positive definite in the case $(A_{C, \varepsilon})$ and is not so in the case $(B_{C, \varepsilon})$, independently of φ . Therefore Theorem 2 is valid.

Under the assumptions of Theorem 3, as follows from the above, for the system (S_ξ, φ_ξ) there exists a function $V(x, \xi) = x^*H(\xi)x$ satisfying the conditions of Lemma 1. Let $\xi_0 \in [0, 1]$, $H^0 = H(\xi_0)$. The derivative $[\dot{V}(x, \xi_0)]_\xi$ along the system (S_ξ, φ_ξ) has the form $[V(x, \xi_0)]_\xi = -W_\xi - F_\xi$, where $W_\xi = -2x^*H^0(P_\xi x + q\xi) - F_\xi$ is a positive definite form for $\xi \in \Delta(\xi_0)$, if $\Delta(\xi_0)$ is a sufficiently small open interval, $\Delta(\xi_0) \ni \xi_0$. Consequently, $V(x, \xi_0)$ satisfies the conditions of Lemma 1 for the system (S_ξ, φ_ξ) when $\xi \in \Delta(\xi_0)$. From the foregoing it follows that for all systems (S_ξ, φ_ξ) , $\xi \in \Delta(\xi_0)$, either the case $(A_{C,\varepsilon})$ or $(B_{C,\varepsilon})$ holds, where C and ε depend on ξ . Choosing from the system $\{\Delta(\xi)\}$, $\xi \in [0, 1]$, a finite covering $\Delta_1, \dots, \Delta_k$, by the usual argument we obtain the assertion of Theorem 3.

We omit the proof of Theorem 4. Assertion I of Theorem 4 is proved by the same scheme as the proof of Theorem 4 in (4).

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Received
8 IX 1968

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