Mathematics

V. A. TOPONOGOV

1969

SovietRxiv

Abstract

Full Text

Mathematics

V. A. TOPONOGOV

EXTREMAL THEOREMS FOR RIEMANNIAN SPACES OF CURVATURE BOUNDED ABOVE

(Presented by Academician A. D. Aleksandrov, 12 V 1968)

Let $R^m(k_0, k_1)$ $(R^m_{k_1})$ be an m-dimensional compact, four-times continuously differentiable, simply connected Riemannian space whose Riemannian curvature K at every point and in every two-dimensional direction satisfies the inequalities $k_0 < K \leq k_1$ $(K \leq k_1)$. In addition, consider the class of Riemannian spaces $R^m_{k_1}(a)$, defined by the conditions:

(1) $R^m_{k_1}(a)$ is a Riemannian space $R^m_{k_1}$ for $k_1 > 0$;

(2) the length of any closed geodesic in $R^m_{k_1}(a)$ is not less than $2\pi/\sqrt{k_1}$.

The class of spaces $R^m_{k_1}(a)$ is nonempty.

As follows from (1–3), the spaces $R^m(0, k_1)$ for $m = 2p$ and the spaces $R^m(1/4\,k_1, k_1)$ for $m = 2p + 1$ belong to the class $R^m_{k_1}(a)$.*

We note that condition (2) is equivalent to conditions (2^*) and (2^{**}):

(2^*) The perimeter of any nondegenerate polygon in $R^m_{k_1}(a)$, composed of geodesics, is not less than $2\pi/\sqrt{k_1}$.

(2^{**}) Any arc of a geodesic in $R^m_{k_1}(a)$ whose length does not exceed $\pi/\sqrt{k_1}$ is shortest.

In this paper the extremal cases of conditions (2), (2^*), and (2^{**}) are considered.

Theorem 1. If in $R^m_{k_1}(a)$ the index of a closed geodesic γ of length $2\pi/\sqrt{k_1}$ is equal to q, then in $R^m_{k_1}(a)$ there exists a $(q+1)$-dimensional totally geodesic surface, isometric to a $(q+1)$-dimensional sphere of radius $1/\sqrt{k_1}$, containing γ.

Theorem 2. If in $R^m_{k_1}(a)$ there exists a nondegenerate polygon γ, composed of geodesics, with perimeter $2\pi/\sqrt{k_1}$, then in $R^m_{k_1}(a)$ there exists a two-dimensional totally geodesic surface, isometric to some polygon on a sphere of radius $1/\sqrt{k_1}$, whose boundary coincides with γ.

Theorem 3. If the diameter of $R^m_{k_1}(a)$ is equal to $\pi/\sqrt{k_1}$, then there exists a number q, equal either to 2, or to 4, or to 8, or to m, such that for every point $P \in R^m_{k_1}(a)$ and every vector λ at the point P there exists a q-dimensional totally geodesic surface $F(P, \lambda)$, isometric to a q-dimensional sphere of radius $1/\sqrt{k_1}$, passing through the point P, whose tangent plane contains λ. Two surfaces $F(P, \lambda_1)$ and $F(P, \lambda_2)$ either coincide or have no common points except P.

From Theorem 1, since the index of any closed geodesic in $R^{2p}(0, k_1)$ is not less than 1 (2), it follows

Theorem 4. If in $R^{2p}(0, k_1)$ there exists a closed geodesic γ of length $2\pi/\sqrt{k_1}$, then in $R^{2p}(0, k_1)$ there exists a two-dimensional totally geodesic surface isometric to a two-dimensional sphere of radius $1/\sqrt{k_1}$, containing γ.

Theorem 4 for $m = 2$ was proved by W. Klingenberg in (5).

Further, from Morse' s comparison theorem one can obtain that the index of any closed geodesic in $R^m(1/4k_1, k_1)$ is not less than $(m-1)$ (6). Therefore, from Theorem 1 it follows that

Theorem 5. *If in* $R^m(1/4k_1, k_1)$ *there exists a closed geodesic of length* $2\pi/\sqrt{k_1}$, *then* $R^m(1/4k_1, k_1)$ *is isometric to an* m*-dimensional sphere of radius* $1/\sqrt{k_1}$.

Theorem 5 was previously proved by the author and, independently, by Tsukamoto (7,8).

Finally, using the last assertion of Theorem 3, one can prove

Theorem 6. *If the diameter of the space* $R^m_{k_1}(a)$, *homeomorphic to a sphere, is equal to* $\pi/\sqrt{k_1}$, *then* $R^m_{k_1}(a)$ *is isometric to an* m*-dimensional sphere of radius* $1/\sqrt{k_1}$.

In fact, the Riemannian spaces satisfying the conditions of Theorem 3 apparently must be isometric to symmetric spaces of rank 1, but the author has not succeeded in proving this assertion for all dimensions, only for $m = 3, 4, 5$.

The following assertion, proved by M. Berger (9), also speaks in favor of our hypothesis.

If the diameter of $R^m(1/4k_1, k_1)$ is equal to $\pi/\sqrt{k_1}$, then $R^m(1/4k_1, k_1)$ is isometric to a symmetric space of rank 1.

Let us indicate the idea of the proof of Theorem 1.

---

* The result announced by the author of this article in (4), asserting that $R^m(0, k_1)$ for $m = 2p + 1$ belongs to the class $R^m_{k_1}(a)$, is incorrect. At the International Congress of Mathematicians in Moscow (1966), V. Klingenberg communicated to the author an example refuting the announced result.

Lemma 1. *Under the conditions and notation of Theorem 1, any two points P and Q of the geodesic γ, separated from each other along γ by the distance $\pi/\sqrt{k_1}$, are conjugate to each other with multiplicity q.*

The proof of Lemma 1 is obtained from condition (2^*) and from the known lemmas of the calculus of variations.

From Lemma 1 it is already not difficult to obtain, by induction,

Lemma 2. *Under the conditions and notation of Theorem 1, there exists an arc σ of the geodesic γ of length greater than $\pi/\sqrt{k_1}$ and a q-parameter family of parallel vector fields ν along σ such that the Riemannian curvature in the two-dimensional directions containing $\dot{\gamma}$ and ν is equal to k_1.*

Now using Lemma 2, for each field ν one can construct a sequence of triangles $\Delta_n(\nu)$ converging to γ and such that, for any n, the perimeter of $\Delta_n(\nu)$ is strictly less than $2\pi/\sqrt{k_1}$. Hence, and from condition (2^{**}), it follows that over each triangle $\Delta_n(\nu)$ one can stretch a cone $K_n(\nu)$ (the cone $K_n(\nu)$ is obtained as the set of shortest curves joining one of the vertices of $\Delta_n(\nu)$ with the points of the opposite side). For the cones $K_n(\nu)$ the following lemmas are true.

Lemma 3. *The Gaussian curvature of $K_n(\nu)$ at each point does not exceed k_1.*

Lemma 3 follows from Synge' s lemma (^{10}).

Lemma 4. *The area of the cone $K_n(\nu)$ does not exceed the area of the triangle $\Delta_n^L(\nu)$, constructed on a sphere of radius $1/\sqrt{k_1}$ with the same side lengths as the triangle $\Delta_n(\nu)$.*

Lemma 4 follows from a theorem of A. D. Aleksandrov (^{11}). From Lemmas 3 and 4 one obtains an upper estimate for the integral curvature of the cone $K_n(\nu)$. On the other hand, from the Gauss-Bonnet theorem one can obtain a lower estimate for the integral curvature of $K_n(\nu)$ in terms of the angles of $\Delta_n(\nu)$. Comparing these estimates, one obtains that the Gaussian curvature of $K_n(\nu)$ is everywhere almost equal to k_1, and the area of $K_n(\nu)$ is almost equal to $2\pi/\sqrt{k_1}$. Passing now to the limit as $n \to \infty$, we obtain that there exists a q-parameter family of surfaces F, isometric to a two-dimensional hemisphere of radius $1/\sqrt{k_1}$, whose boundary coincides with γ.

Now it is no longer difficult to show that the totality of all surfaces of this family forms a $(q+1)$-dimensional surface F^{q+1}, isometric

$(q+1)$-dimensional sphere of radius $1/\sqrt{k_1}$. Finally, from (2^{**}) and the preceding it is not difficult to obtain that F^{q+1} is a totally geodesic surface in $R^n_{k_1}(a)$.

The proofs of Theorems 2 and 3 are obtained by applying arguments analogous to those given above; only in the proof of Theorem 3 it is necessary to use certain topological considerations, in particular, the theorem of F. Browder (^{12}).

Institute of Mathematics Siberian Branch of the Academy of Sciences of the USSR

Received 19 IV 1968

REFERENCES

1. A. V. Pogorelov, *Matem. sborn.*, **18** (**60**), 181 (1964).
2. W. Klingenberg, *Ann. Math.*, **69**, 654 (1959).
3. W. Klingenberg, *Comm. Math. Helv.*, **34**, 47 (1961).
4. V. A. Toponogov, DAN, **154**, No. 5, 1047 (1964).
5. W. Klingenberg, *Comm. Math. Helv.*, **34**, f. 1 (1960).
6. M. Morse, *The Calc. of Variat. in the Large*, 1934.
7. I. Tsukamoto, *Tohoku Math. J.*, **18**, No. 2, 138 (1966).
8. V. A. Toponogov, *Sibirsk. matem. zhurn.*, **8**, No. 5 (1967).
9. M. Berger, *Ann. Scuola Norm. Sup. Pisa*, Ser. III, **14**, f. II (1960).
10. J. L. Synge, *Proc. London Math. Soc.*, **25**, 274 (1926).
11. A. D. Aleksandrov, *Akad. wiss. Forsch. Math.*, No. 1, 33 (1957).
12. W. Browder, *Bull. Am. Math. Soc.*, **68**, No. 3, 202 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.