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# CONES WITH A TRANSITIVE GROUP

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## Abstract

## Full Text

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*MATHEMATICS*

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# CONES WITH A TRANSITIVE GROUP

1. Let  $C$  be a cone in  $n$ -dimensional affine space  $A_n$  ( $n \geq 3$ ), i.e., a set of points consisting of half-lines emanating from one point  $O$ —the vertex of the cone, and let  $C_X$  be the cone with vertex  $X$ , obtained from  $C$  by a parallel translation. Let  $f$  be a one-to-one mapping of  $A_n$  onto itself such that for every  $X$ ,  $f(C_X) = C_{f(X)}$  and  $f(O) = O$ , so that  $f(C) = C$ . We denote by  $F_C$  the group of all such mappings.

We consider cones  $C$  with transitive  $F_C$ , i.e., with the condition:

(A) For any points  $Y, Z \in C$ , different from the vertex, there exists a mapping  $f$  such that  $f(Y) = Z$ .

Since all cones  $C_X$  are equal and parallel, by adjoining to the mappings  $f$  parallel translations we obtain mappings  $g$  which, in general, preserve the system of cones  $C_X$ :  $g(C_X) = C_{g(X)}$ . Then condition (A) takes the form:

(A') The system of cones  $C_X$  is such that for any points  $X_1, X_2$  and  $Y_1 \in C_{X_1}, Y_2 \in C_{X_2}$  ( $Y_i \neq X_i$ ) there exists

If  $h$  is an affine mapping, then from  $h(O) = O$ ,  $h(C) = C$  it follows that  $h(C_X) = C_{h(X)}$ . Therefore, if one restricts oneself in advance to affine mappings, the question concerns cones admitting a transitive group of mappings onto themselves. Such convex cones have been considered, in particular, in <sup>(1, 2)</sup>. One such cone is, in particular, the elliptic cone, i.e., one representable in suitable coordinates by the equation  $x_n = \sqrt{x_1^2 + \dots + x_{n-1}^2}$ . Its group of affine mappings onto itself is the Lorentz group with the inclusion of homotheties with respect to the vertex. At the same time, the following holds.

**Theorem 1.** *If a cone satisfying condition (A) is no less than an  $(n - 1)$ -dimensional closed set, contained in a half-space and different from a plane, then it is elliptic and the group  $F_C$  of all its mappings  $f$  is the Lorentz group.*

We say that the vertex of the cone  $C$  is sharp if it has a supporting plane containing no points of its closure  $\bar{C}$  except the vertex.

**Theorem 2.** *Let a cone  $C$  satisfying condition (A) have a sharp vertex  $O$  and divide the space ( $A_n \setminus C$  is not connected). Suppose, moreover, that at least one of the sets  $C \setminus \{O\}$ ,  $A_n \setminus C$  has no more than a countable number of connected components. Then the cone  $C$  is elliptic and  $F_C$  is the Lorentz group.*

The condition on the number of components is possibly superfluous; the others are not. If the condition of sharpness of the vertex is replaced by the condition that  $C$  is contained in a half-space, then new possibilities arise, not counting the trivial one when  $C$  is a plane.

2. An elliptic cone together with its reflection symmetric with respect to the vertex forms a double elliptic cone. It bounds three open cones: two mutually symmetric “inner” ones and one “outer” one. The group of its affine mappings onto itself is transitive on it and is the “doubled” Lorentz group, i.e., supplemented by symmetries at the vertex.

**Theorem 3.** *Let a cone  $C$ , different from all of  $A_n$ , contain an  $(n - 1)$ -dimensional plane  $P$  and satisfy condition (A), in which, however,*

mappings  $f$  are subject to the following additional restriction. If  $R_1, R_2$  are half-spaces bounded by the plane  $P$ , then  $f$  maps each of the sets  $K_i = (A_n \setminus C) \cap R_i$ ,  $i = 1, 2$ , onto itself.

Then there are only three possibilities: (1)  $C$  is an exterior elliptic cone, and  $F_C$  is the doubled Lorentz group (all mappings  $f$  without additional restrictions belong to  $F_C$ ); (2)  $C \setminus \{O\}$  is obtained by deleting from  $A_n$  some line passing through  $O$  and not lying in the plane  $P$ ; (3)  $C = P$ .

In the last two cases the mappings  $f$ , obviously, may be quite arbitrary.

Since, by assumption,  $C \neq A_n$ , at least one of the sets  $K_i$  is nonempty, and then  $K_i \cup \{O\}$  is, obviously, a cone with vertex  $O$ .

From Theorem 3 there follow the two following theorems.

**Theorem 4.** *If a cone satisfying the assumptions of Theorem 3:  $C \neq A_n$ ,  $C \supset P$ , satisfies condition (A) with continuous mappings  $f$ , then the conclusion of Theorem 3 holds for it.*

**Theorem 5.** *Let the cone  $C$  satisfying the assumptions of Theorem 3 be such that at least one (nonempty) set  $K_1$  or  $K_2$  is convex. Then, if  $C$  satisfies condition (A) without additional restrictions on the mappings  $f$ , the conclusion of Theorem 3 also holds.*

It seems probable that, in general, any additional restrictions whatever on the mappings  $f$  or on the sets  $K_1, K_2$  are superfluous.

3. The case opposite to that appearing in Theorems 3-5 is represented by a cone with an acute vertex, containing interior points. However, such a cone satisfying condition (A) need not be internally elliptic. The simplest example is the interior of an  $n$ -hedral angle. Nevertheless, the following holds.

**Theorem 6.** Let a cone  $C$  satisfying condition (A) have an acute vertex and contain interior points. Then it is convex and open ( $C \setminus \{O\}$  is an open set). It will be internally elliptic and  $F_C$  a Lorentz group if its boundary  $\partial C$  satisfies one of the following conditions: (1)  $\partial C$  is smooth (apart, of course, from the vertex); (2) the section of  $\partial C$  by a plane intersecting all generators contains at least one point at which this section has an osculating paraboloid (not degenerating into a cylinder).

Without the indicated additional conditions one can assert the following.

**Theorem 7.** Let  $C$  be a cone satisfying the initial assumptions of Theorem 6. If its closure  $\bar{C}$  is not the Cartesian product of a half-line by an  $(n-1)$ -dimensional cone, then all its mappings  $f$  are affine. In the contrary case among them there certainly are mappings distinct from affine ones.

Namely, let  $\bar{C} = L \times C'$ , where  $L$  is a half-line, and the cone  $C'$  lies in the plane  $P$ . Let  $h$  be a homeomorphism of the line  $\bar{L} \supset L$  onto itself such that  $h(L) = L$ , and let  $g$  be such a mapping of  $A_n$  onto itself under which, for every point  $X \in \bar{L}$ , the plane  $P_X \parallel P$  passing through it undergoes the parallel translation  $X \rightarrow h(X)$ . Then  $g$  is a mapping  $f$ . At the same time, if  $\bar{C} = L_1 \times \dots \times L_m \times C''$  and  $C' \neq L \times C''$ , then every mapping  $f$  is a combination of an affine mapping of  $C$  onto itself and mappings  $g$  corresponding to the half-lines  $L_i$ .

Open convex cones with an acute vertex, with transitive groups of affine mappings, were studied in <sup>(1)</sup>. The algebraic theory developed there is sufficiently complicated, which, however, corresponds to the variety of such cones in spaces of large dimension. In this connection we note a special result which can be obtained by direct geometric means.

**Theorem 8.** In  $A_3$  and  $A_4$  a cone  $C$  satisfying the initial assumptions of Theorem 6 is either internally elliptic (and then  $F_C$  is a Lorentz group), or the interior of a trihedral and, respectively, tetrahedral angle, or,

in  $A_4$ ,  $C$  is the Cartesian product of a three-dimensional interior elliptic cone and a half-line (of course, with this half-line itself removed).

4. The results presented are connected with the foundations of relativity theory. The cone  $C$  in Theorems 1 and 2 may be understood as the light cone—the set of points of space-time (or events) reached by light from a flash at  $O$ . Accordingly, the cone  $C$  in Theorems 3-5 is formed by points of space-time to which no influences from the event  $O$  at the vertex of the cone  $C$  reach, and events at which also do not act on  $O$ . In short, the cone  $C$  is formed by events  $X$  mutually not acting with  $O$ . In relativity theory the intervals  $OX$  are space-like. Finally, the cone  $C$  in Theorems 6-8 is formed by points of space-time to which “non-light” influences issuing from  $O$  may reach—motions of bodies of nonzero mass, in particular, motions by inertia.

The condition ( $A'$ ) of the existence of a group  $G_C$ , transitive with respect to pairs of points  $X_1, Y_1 \subset C_{x_1}$  and  $X_2, Y_2 \subset C_{x_2}$ , expresses, in the language

accepted in physics, the equivalence of such pairs. Theorems 1 and 2 show that such an equivalence of pairs of events connected with the propagation of light entails the geometry of relativity theory. The condition (A) itself may here be understood as expressing the isotropy of the propagation of light from any flash  $O$ .

Theorems 3–5 give conditions under which the equivalence of pairs of mutually non-acting events also entails the geometry of relativity theory. From the physical point of view these conditions are quite natural. The condition of Theorem 3 excludes the interchange of past and future—of events acting on  $O$ , with events on which  $O$  acts. The continuity condition in Theorem 4 needs no comment. The convexity condition of the cone  $K_1$  (or  $K_2$ ) in Theorem 5 means the transitivity of the succession of events.

In contrast to these conclusions, what was said in item 3 shows that the equivalence of pairs of events connected by non-light influences entails the geometry of relativity theory only under additional conditions on the corresponding cone  $C$ , indicated in Theorem 6. Theorem 8 indicates those possibilities which are obtained in the case of four-dimensional space-time when any additional conditions are removed. These are the possibilities in which equivalence of inertial motions can occur, i.e. the Galilean principle.

5. The proof of the theorems stated rests above all on the results of [3]. From them it is easy to conclude that, under the conditions of Theorem 1, the cone  $C$  is the boundary of a solid convex cone with a sharp vertex, and all its images  $f$  are affine. Therefore there exists a plane intersecting all generators of the cone  $C$ , and the section is a closed convex surface  $S$  in  $A_{n-1}$ , admitting a transitive group  $G$  of projective transformations onto itself ( $A_{n-1}$  is completed to the projective space  $P_{n-1}$ ).

The group  $G$  may be considered closed (extending it, if necessary, to its closure) in the full projective group. Then it is a Lie group [4]. Therefore the surface  $S$  is regular and, consequently, has everywhere positive curvature.

Let  $D$  be the domain bounded by  $S$ . We distinguish two cases: (1) in  $D$  there is a point  $\mathcal{A}$  such that the set  $G(\mathcal{A})$  has limit points on  $S$ ; (2) there are no such points in  $D$ .

In the second case one can verify that if  $\mathcal{A}_1, \dots, \mathcal{A}_{n+1}$  are points in general position in  $D$ , then for every sequence of mappings  $g_i \in G$  the points  $\mathcal{B}_r$ , serving as limits of subsequences  $g_{i_k}(\mathcal{A}_r)$ , will also be in general position. Therefore there exists a projective mapping  $g$  taking the points  $\mathcal{A}_r$  to  $\mathcal{B}_r$ , so that  $g = \lim g_{i_k}$  and  $g \in G$ . Hence  $G$  is compact and, consequently, is similar to a subgroup of the orthogonal group, ot-

whence it is easy to conclude that the surface  $S$  is represented in the corresponding coordinates as a sphere, i.e., is an ellipsoid.

Now suppose that case (1) holds, so that there is a point  $\mathcal{A} \in D$  and mappings  $g_i \in G$  such that  $\mathcal{A}_i = g_i(\mathcal{A}) \rightarrow \mathcal{B} \in S$ . Through  $\mathcal{A}_i$  draw a plane  $P_i$  parallel to

the tangent plane at the point nearest to  $\mathcal{A}_i$  on  $S$  (for simplicity we regard the space as Euclidean). This plane cuts off from  $S$  a cap  $S_i$ , close to a parabolic one, i.e., to an ellipsoid in the projective sense. The transformation  $g_i^{-1}$ , taking  $\mathcal{A}_i$  to  $\mathcal{A}$ , takes the plane  $P_i$  to a plane  $P$  passing through  $\mathcal{A}$ , and the cap  $S_i$  to a part  $S_i^*$  of the surface  $S$  cut off by  $P$ . Closeness to a part of an ellipsoid is preserved under a projective mapping. Therefore, in the limit, as  $i \rightarrow \infty$ , we are convinced that on the part cut off by some plane passing through  $\mathcal{A}$ , the surface  $S$  is an ellipsoid. Then, by the transitivity of the group  $G$ , the whole of  $S$  is an ellipsoid. Theorem 1 is proved.

The proof of Theorem 2, and also of Theorem 3, consists mainly in reducing them to Theorem 1. Theorems 4 and 5 are easily derived from Theorem 3. Theorem 7 follows directly from the results of paper <sup>(3)</sup>. Theorems 6 and 8 are proved by applying considerations analogous to those used in the proof of Theorem 1.

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*Note: Figure translations are in progress. See original paper for figures.*

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