A Nonhomogeneous Semi-Infinite Atmosphere with Pure Scattering

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Abstract

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A Nonhomogeneous Semi-Infinite Atmosphere with Pure Scattering

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In the theory of radiative transfer it is usually assumed that the scattering indicatrix $\chi(\gamma)$ and the probability of survival of a quantum λ do not depend on the optical depth τ, i.e., it is assumed that the transfer of radiation takes place in a homogeneous medium. However, in planetary and stellar atmospheres this condition is never strictly satisfied. Therefore let us consider a plane semi-infinite atmosphere in which $\lambda = \lambda(\tau)$ and $\chi(\gamma) = \chi(\gamma, \tau)$, i.e., the probability of survival of a quantum and the scattering indicatrix are arbitrary functions of the optical depth τ.

Let

$$\chi(\gamma, \tau) = \sum_{i=1}^{n} x_i(\tau) P_i(\cos\gamma), \tag{1}$$

where $P_n(\cos\gamma)$ is a Legendre polynomial. Then the brightness coefficient of the diffusely reflected radiation is

$$\rho(\eta, \xi, \varphi - \varphi_0) = \sum_{m=0}^{n} r_m(\eta, \xi) \cos m(\varphi - \varphi_0). \tag{2}$$

Here $\arccos\xi$ and $\arccos\eta$ are, respectively, the angles of incidence and reflection of light at the upper boundary of the atmosphere, and φ_0 and φ are the azimuths of the incident and reflected rays. In addition,

$$r_m(\eta, \xi) = \frac{1}{\delta_m} \sum_{k=m}^{n} (-1)^{k+m} \int_0^\infty \mu_k^m(\tau) \varphi_k^m(\xi, \tau) \varphi_k^m(\eta, \tau) e^{-\tau\left(\frac{1}{\eta}+\frac{1}{\xi}\right)} \frac{d\tau}{\eta\xi}, \tag{3}$$

where

$$\varphi_k^m(\eta,\tau) = P_k^m(\eta) + \sum_{i=m}^{n} (-1)^{i+k} \int_\tau^\infty \mu_i^m(t)\varphi_i^m(\eta,t)e^{-\frac{t-\tau}{\eta}}\,dt \times \int_0^1 \varphi_i^m(\eta',t)P_k^m(\eta')e^{-\frac{t-\tau}{\eta'}}\frac{d\eta'}{\eta'}, \tag{4}$$

$$\delta_m - \begin{cases} 2, & m = 0, \\ 1, & m = 1; \end{cases} \tag{5}$$

$$\mu_i^m(\tau) = \frac{1}{2}\lambda(\tau)x_i(\tau)\frac{(i-m)!}{(i+m)!}. \tag{6}$$

Formulas (3) and (4) for a spherical scattering indicatrix were first obtained by V. V. Sobolev ([1]), and for anisotropic scattering they were found by the author ([2]).

The function $\varphi_k^m(\eta,\tau)$ is the function φ_k^m of V. A. Ambartsumian for a nonhomogeneous semi-infinite layer on top of which a layer of optical thickness τ has been placed. Such a layer we shall subsequently call

for brevity, call truncated. Consequently, if the medium is homogeneous, then the function $\varphi_k^m(\eta,\tau)$ does not depend on τ, and from formulas (3) and (4), after integration, the well-known formulas of V. A. Ambartsumian are easily obtained ([3]).

The function $\varphi_k^m(\tau,\eta)$ can also be written in the following form:

$$\varphi_k^m(\eta,\tau) = P_k^m(\eta) + (-1)^{k+m}\delta_m\eta\int_0^1 r_m(\eta,\eta',\tau)P_k^m(\eta')\,d\eta', \tag{7}$$

Here $r_m(\eta,\zeta,\tau)$ is a component of the brightness coefficient (2) for the truncated layer.

As is known ([4]), the plane albedo of a semi-infinite medium is determined by the formula

$$A(\zeta) = 2\int_0^1 r_0(\eta,\zeta)\eta\,d\eta \tag{8}$$

or, for a truncated layer,

$$A(\zeta,\tau) = 2\int_0^1 r_0(\eta,\zeta,\tau)\eta\,d\eta. \tag{9}$$

From formulas (5), (7), and (9) we find

$$A(\zeta,\tau) = 1 - \frac{1}{\zeta}\varphi_1^0(\zeta,\tau). \tag{10}$$

Let pure scattering occur in the atmosphere at all optical depths, i.e., set $\lambda(\tau) = 1$. Obviously, in this case the plane albedo is

$$A(\zeta,\tau) = 1. \tag{11}$$

Consequently, from (10) and (11) we find

$$\varphi_1^0(\zeta,\tau) = 0 \tag{12}$$

or

$$2\int_0^1 r_0(\eta,\zeta,\tau)\eta\,d\eta = 1. \tag{13}$$

If we denote

$$a_{i0}^0(\tau) = \int_0^1 \varphi_i^0(\eta,\tau)\,d\eta, \tag{14}$$

then from (7) and (13) we obtain

$$a_{00}^0(\tau) = 2; \quad a_{i0}^0 = 0 \quad \text{for } i = 1,2,3,\dots,n. \tag{15}$$

Formulas (12), (13), and (15) for a homogeneous medium were known earlier (see, for example, ([5])). As we see, these formulas are also valid in the case when the scattering indicatrix is an arbitrary function of optical depth.

Let us consider two special cases of the atmosphere for $\lambda = 1$.

1. Let the scattering indicatrix have the form

$$\chi(\gamma,\tau) = 1 + x_1(\tau)\cos\gamma, \tag{16}$$

i.e., in formula (1) $n = 1$. Let us find an expression for the brightness coefficient (2) averaged over azimuth, i.e., the quantity $r_0(\eta,\zeta,\tau)$. Taking formula (12) into account, we come to the conclusion that in the case under consideration the function $\varphi_0^0(\eta,\tau) = \varphi(\eta)$, i.e., it does not depend on τ. Then from relations (3)–(6) ···

we obtain

$$r_0(\eta,\xi) = -\frac{1}{4}\,\frac{\varphi(\eta)\varphi(\xi)}{\eta+\xi}, \tag{17}$$

where

$$\varphi(\eta) = 1 + \frac{1}{2}\eta\varphi(\eta)\int_0^1 \frac{\varphi(\eta')}{\eta+\eta'}\,d\eta'. \tag{18}$$

Thus, the azimuth-averaged brightness coefficient for pure scattering and phase function (16) coincides completely with the brightness coefficient for isotropic scattering. This conclusion, in the special case $x_1(\tau) = \text{const}$, was drawn by V. A. Ambartsumian [3].

2. Let the medium consist of two layers. In the upper layer of optical thickness τ_0, scattering occurs with a spherical scattering phase function, while in the lower layer it occurs with scattering phase function (16), with $x_1(\tau) = x_1 = \text{const}$. In this case we have

$$\rho(\eta,\xi,\varphi-\varphi_0) = r_0(\eta,\xi) + r_1(\eta,\xi)\cos(\varphi-\varphi_0), \tag{19}$$

where the function $r_0(\eta,\xi)$, on the basis of what was set forth above, is determined by expression (17). For the quantity $r_1(\eta,\xi)$, from formulas (2)–(6) and (12) we obtain

$$r_1(\eta,\xi) = \frac{x_1}{4}\,\frac{\varphi_1^1(\eta)\varphi_1^1(\xi)}{\eta+\xi}\,e^{-\tau_0\left(\frac{1}{\eta}+\frac{1}{\xi}\right)}, \tag{20}$$

where the function $\varphi_1^1(\eta)$ is found from the integral equation obtained by V. A. Ambartsumian [3]:

$$\varphi_1^1(\eta) = \eta - \frac{1}{2}\eta\varphi_0^0(\eta)\int_0^1 \frac{\varphi_0^0(\eta')}{\eta+\eta'}\,\eta'\,d\eta' + \frac{1}{2}x_1\eta\varphi_1^0(\eta)\int_0^1 \frac{\varphi_1^0(\eta')}{\eta+\eta'}\,\eta'\,d\eta', \tag{21}$$

with $\varphi_0^0(\eta) = \varphi(\eta)$. Thus, the entire dependence on the optical thickness of the upper layer has been reduced to the appearance of an exponential factor in expression (20). This is quite natural, since in the upper layer the scattering phase function is spherical.

It is known that exact analytic expressions exist for the functions $\varphi(\eta)$ and $\varphi_1^1(\eta)$. Consequently, as is clear from the examples given above, certain particular problems of radiative transfer in a semi-infinite inhomogeneous medium admit an exact analytic solution.

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Note: Figure translations are in progress. See original paper for figures.