ON SUBELLIPTIC PSEUDODIFFERENTIAL OPERATORS

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Abstract

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1. In this note exact necessary and sufficient conditions of an algebraic character are given under which a pseudodifferential operator is subelliptic in a domain $\Omega \subset \mathbb{R}^n$, i.e., such that for every compact subset $K \subset \Omega$ there exists a constant $C = C(K)$ and the inequality

$$|u|_s \leqslant C(|Pu|_{s-m+\delta} + |u|_{s-1}), \qquad u \in C_0^\infty(K) \tag{1}$$

is satisfied for $0 \leq \delta < 1$ ($|\ |_s$ is the norm in the Sobolev–Slobodetskii space $\mathcal{H}_s$).

The study of this class of operators was begun by L. Hörmander in the work ([1]), where conditions on the symbol $p^0(x,\xi)$ were indicated under which inequality (1) holds with $\delta = 1/2$, and it was proved that if (1) is satisfied for $0 < \delta < 1/2$, then the operator P is elliptic, i.e., one may take $\delta = 0$. In the works ([2],[3]) conditions of implicit form were obtained which are necessary and sufficient for the subellipticity of operators. In the work ([3]) operators were found for which (1) is satisfied with $\delta = (k-1)/k$, and a conjecture was put forward on the absence of operators for which (1) is satisfied with intermediate values of δ (and there is no better estimate). A complete proof of this conjecture is obtained only in the present work. In the work ([4]) conditions were obtained under which (1) is satisfied with $\delta = 2/3$; in the work ([6]) inequality (1) was studied in connection with the class of so-called nondegenerate operators; in the work ([7]) certain conditions necessary for estimates of the form (1) with arbitrary real δ to hold were obtained. In doing so, the results of the work ([5]) on a priori estimates for first-order differential operators were substantially used.

It should be noted that the significance of the results obtained goes beyond the question of the validity of certain estimates. The conditions obtained here make it possible to solve completely questions of solvability (local or global), smoothness, hypoellipticity, etc., for one class of pseudodifferential operators, in which these questions are naturally decided in terms only of the principal part of the symbol. This class is defined by the following condition: for each point $(x,\xi) \in \Omega \times \{\mathbb{R}^n \setminus 0\}$ in the Lie algebra generated by the operators P and

P^*, one can indicate an operator whose leading part of the symbol is nonzero at this point.

2. Let the operator P^* be formally adjoint to the operator P. Consider all possible commutators C_I composed of the operators P and P^*. Here I is a finite sequence of ones and twos, so that if $I = (i_1, i_2, \dots, i_s)$, where $i_j = 1$ or 2, then

   $$C_I = \operatorname{ad} P_{i_1} \operatorname{ad} P_{i_2} \cdots \operatorname{ad} P_{i_{s-1}} \dots, P_{i_s},$$

   where $P_1 = P$ and $P_2 = P^*$. (For example,

   $$C_{21} = [P^*, P] \equiv P^*P - PP^*, \quad C_{221} = [P^*, C_{21}],$$

   etc.) Put $|I| = s - 1$. It is obvious that the order of the operator C_I does not exceed $m + |I|(m-1)$, if m is the order of the operator P. Let $c_I^0(x,\xi)$ be the principal part of the symbol of the operator C_I (positively homogeneous in ξ of degree $m + |I|(m-1)$). The indicated

the class of operators is determined by the following condition: at each characteristic point (x,ξ) at least one of the functions $c_I^0(x,\xi)$ is different from zero. Put $k(x,\xi) = |I_0|$, if $p^0(x,\xi) = 0$, $c_I^0(x,\xi) = 0$ for $|I| < |I_0|$, but $c_{I_0}(x,\xi) \neq 0$. If $p^0(x,\xi) \neq 0$, we put $k(x,\xi) = 0$.

Let us also note that, for every subelliptic operator defined on a compact manifold, one can indicate two scales of function spaces such that the operator P realizes an almost isomorphic mapping (i.e. one for which $\dim\ker P < \infty$, $\dim\operatorname{coker} P < \infty$) of each space of one scale onto some space of the other.

We record the following simple assertion:

Lemma. If at some point $(x^0,\xi^0) \in \Omega \times \{\mathbb{R}^n \setminus 0\}$ we have $0 < k(x^0,\xi^0) < \infty$, then $\operatorname{grad}_{x,\xi} p^0(x^0,\xi^0) \neq 0$.

For the formulation of the main result we shall need the following notion (cf. with (10)). We shall call a bicharacteristic corresponding to the function $\operatorname{Re} p^0(x,\xi)$ any curve $x = x(t)$, $\xi = \xi(t)$ in the space (x,ξ) satisfying the system of differential equations

$$\frac{dx_j}{dt} = \frac{\partial \operatorname{Re} p^0(x,\xi)}{\partial \xi_j}, \qquad \frac{d\xi_j}{dt} = -\frac{\partial \operatorname{Re} p^0(x,\xi)}{\partial x_j}, \qquad j = 1, \dots, n.$$

It is easy to see that if $\operatorname{Re} p^0(x(0),\xi(0)) = 0$, then $\operatorname{Re} p^0(x(t),\xi(t)) \equiv 0$. In this case we shall call the bicharacteristic a null one. Bicharacteristics corresponding to the function $\operatorname{Im} p^0(x,\xi)$ are defined analogously.

3. Theorem 1. The operator P is subelliptic and inequality (1) holds for $\delta = k/(k+1)$ if and only if the following conditions are fulfilled:

1°. If $p^0(x_0, \xi_0) = 0$, $x_0 \in \Omega$, $\xi_0 = 0$, and the vectors $\operatorname{Re}\operatorname{grad}_{x,\xi} p^0(x_0, \xi_0)$, $\operatorname{Im}\operatorname{grad}_{x,\xi} p^0(x_0, \xi_0)$ are noncollinear, then at each characteristic point from some neighborhood of the point (x_0, ξ_0) in the direct product $\Omega \times \mathbb{R}^n$ the function $c^0_{21}(x, \xi)$ is nonnegative.

2°. If $p^0(x_0, \xi_0) = 0$, $x_0 \in \Omega$, $\xi_0 \neq 0$, the vectors $\operatorname{Re}\operatorname{grad}_{x,\xi} p^0(x_0, \xi_0)$, $\operatorname{Im}\operatorname{grad}_{x,\xi} p^0(x_0, \xi_0)$ are collinear, and the number $k(x_0, \xi_0)$ is even, then in some neighborhood of the point (x_0, ξ_0) the function $\operatorname{Re} p^0(x, \xi)$ does not change sign along each null bicharacteristic corresponding to $\operatorname{Im} p^0(x, \xi)$, and the function $\operatorname{Im} p^0(x, \xi)$ does not change sign along each null bicharacteristic corresponding to $\operatorname{Re} p^0(x, \xi)$.

3°. If $p^0(x_0, \xi_0) = 0$, $x_0 \in \Omega$, $\xi_0 \neq 0$, the vectors $\operatorname{Re}\operatorname{grad}_{x,\xi} p^0(x_0, \xi_0)$, $\operatorname{Im}\operatorname{grad}_{x,\xi} p^0(x_0, \xi_0)$ are collinear, and the number $k(x_0, \xi_0)$ is odd, then in some neighborhood of the point (x_0, ξ_0) the function $\operatorname{Re} p^0(x, \xi)$ changes sign along each null bicharacteristic corresponding to $\operatorname{Im} p^0(x, \xi)$ at most once, and the function $\operatorname{Im} p^0(x, \xi)$ changes sign along each null bicharacteristic corresponding to $\operatorname{Re} p^0(x, \xi)$ at most once. Moreover, if $\arg\operatorname{grad}_{x,\xi} p^0(x_0, \xi_0) = \varphi$ (i.e. the vector $e^{-i\varphi}\operatorname{grad}_{x,\xi} p^0(x_0, \xi_0)$ is real) and $Q = P \exp i\left(\frac{\pi}{2} - \varphi\right)$, then the principal part of the symbol of the operator $(\operatorname{ad} Q^*)^{k(x_0,\xi_0)} Q$ of order $m + k(x_0, \xi_0)(m-1)$ is positive in this neighborhood.

4°. $k(x, \xi) \leq k$ for all $x \in \Omega$, $\xi \neq 0$.

If $k(x_0, \xi_0) = k$, where $x_0 \in K$, $\xi_0 \neq 0$, then for the operator P an estimate (1) is impossible for $\delta < k/(k+1)$.

Theorem 1 was proved earlier for a number of special cases: in the case $k(x, \xi) \leq 1$—in the paper (1), in the case $k(x, \xi) \leq 2$—in (4), and in the case when the vectors $\operatorname{Re}\operatorname{grad}_{x,\xi} p^0(x, \xi)$ and $\operatorname{Im}\operatorname{grad}_{x,\xi} p^0(x, \xi)$ are noncollinear at characteristic points—in the paper (6). Its proof in the general case is based on the results obtained in (1, 3, 5–8).

4. Theorem 2. If at some characteristic point (x^0, ξ^0) we have $k(x^0, \xi^0) < \infty$, but conditions 1°-3° of Theorem 1 are not fulfilled, then inequality (1) cannot hold for any real δ.

This theorem was, in essence, proved by us earlier in connection with the study of conditions for local solvability of pseudodifferential equations (see (7)).

Theorem 3. *Suppose that for the operator P the function $k(x, \xi)$ is finite at every point of $\Omega \times \{\mathbb{R}^n \setminus 0\}$. In order that the operator P be hypoelliptic, it is necessary and sufficient that it be subelliptic.*

Applying the closed graph theorem in the usual way, one can derive Theorem 3 from Theorems 1 and 2.

A simple consequence of Theorem 1 is the following.

Theorem 4. *If inequality (1) is satisfied for all functions* $u \in C_0^\infty(K)$ *with some* $\delta < 1$ *and* $(k-1)/k < \delta < k/(k+1)$, k *an integer, then there exists a constant* $C_1(K)$ *such that*

$$|u|_s \leq C_1(K)(|Pu|_{s-m+(k-1)/k} + |u|_{s-1}), \quad u \in C_0^\infty(K).$$

In some cases it is useful to use function spaces different from H_s. In these cases the following result is convenient.

Theorem 5. *If the operator* P *is subelliptic, then for every compact subset* K *of* Ω *and for arbitrary real* s *and* t *one can indicate a constant* $C = C(K, s, t)$ *such that*

$$|P^* u|_s \leq C(|Pu|_s + |u|_t), \quad u \in C_0^\infty(K),$$

where P^* is the operator formally adjoint to the operator P.

5. From Theorem 1 it follows, in particular, that for the solution of the problem with oblique derivative for an elliptic equation of second order one can obtain sharper a priori estimates than in (9) (and, correspondingly, refine the theorems on smoothness, solvability, etc.). Namely, if the order of contact of the field with the boundary does not exceed k and an estimate of the solution with $\delta = 1$ holds (see (9)), then in fact an estimate with $\delta = k/(k+1)$ is valid (see also (3)).

We note that the results of the present note are valid for more general classes of pseudodifferential operators introduced by L. Hörmander in (2).

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Note: Figure translations are in progress. See original paper for figures.