ON THE QUESTION OF THE DEPENDENCE OF HOLOMORPHIC DIFFERENTIALS ON THE MODULI OF RIEMANN SURFACES

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 517.543

MATHEMATICS

S. L. KRUSHKAL'

ON THE QUESTION OF THE DEPENDENCE OF HOLOMORPHIC DIFFERENTIALS ON THE MODULI OF RIEMANN SURFACES

(Presented by Academician M. A. Lavrent' ev, 17 IV 1969)

1. Along with the study by many authors of the nature of the parameters determining the conformal structure of a Riemann surface—the moduli—the question of the character of the dependence on these moduli of various quantities connected with the surface has also been intensively investigated: differentials, Weierstrass points, etc. Various relations have been obtained here, mainly in terms of the theory of theta functions, which are very complicated. The introduction of a complex-analytic structure in the corresponding space of Riemann surfaces gives another approach to the solution of this question (see, for example, ([2,3,6])).

In this note we formulate a theorem that gives a concrete basis in the vector space of holomorphic quadratic differentials (automorphic forms of weight (-4)) on every closed surface of genus $g > 1$, depending holomorphically on the moduli, and thereby gives, in a certain sense, an explicit form of the holomorphic continuation of such differentials into moduli space.

2. Let U_1 be the disk $|z| < 1$; U_2 the domain $1 < |z| \leq \infty$; Γ a fixed Fuchsian group of the first kind such that $S = U_2/\Gamma$ is a closed Riemann surface of genus $g > 1$, with Γ chosen in accordance with what is stated in ([4]), § 2. On the surface S we fix a class of homeomorphisms of S onto itself, for example, one containing the identity mapping.

Let $B_1(\Gamma)$ be the set of functions $\mu(z)$, measurable in the z-plane, such that $\mu(z)\, d\bar{z}/dz$ is invariant with respect to Γ, $\mu(z) = 0$ for $z \in U_1$, and $\|\mu\|_{L_\infty(U_2)} < 1$. For each $\mu \in B_1(\Gamma)$ there exists a unique quasiconformal homeomorphism $w = f_\mu(z)$ of the extended complex z-plane onto itself satisfying the Beltrami equation $w_{\bar{z}} = \mu w_z$ and normalized by the conditions $w(0) = 0,\ w'(0) = 1,\ w(1) = 1$. Then $\Gamma^\mu = f_\mu \Gamma f_\mu^{-1}$ is a quasifuchsian group of fractional-linear mappings of the w-plane, and $S^\mu = f_\mu(U_2)/\Gamma^\mu = f_\mu(S)$ is a closed Riemann surface of genus $g > 1$. Letting μ run through the entire set $B_1(\Gamma)$, we obtain all points $[S^\mu]$ of the Teichmüller space $T(S) = T_g$ (see, for example, ([1,3,5,7])). We shall call functions μ and ν equivalent if they determine one and the same point of the space T_g. Using the results of ([1,3]), we obtain that functions μ and ν from $B_1(\Gamma)$ are equivalent if and only if $f_\mu(z) = f_\nu(z)$ for $z \in \overline{U_1}$.

As moduli of the points $[S^\mu]$ of the space $T(S)$ one can take the corresponding $3g$ coefficients among the coefficients $a = (a_2, a_3, \dots, a_{3g-1})$ of the expansions

$$f_\mu(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in U_1,$$

which fill a bounded domain D_{3g-3} on an analytic surface in $\mathbb{C}^{3g-2}$. These coefficients depend only on the equivalence classes $[\mu]$. Moreover, in each equivalence class $[\mu]$ there exists a function $\mu_a(z) \in B_1(\Gamma)$, depending holomorphically on these a as an element of $L_\infty(U_2)$, and the mapping $\tau : \mu_a \to a$ is biho-

locally homeomorphic homeomorphism from $L_\infty(U_2)$ to C^{3g-2}, generating a well-defined biholomorphic homeomorphism of the space $T(S)$ onto the domain D_{3g-3}. We shall denote the corresponding $3g\hat{}$ coefficients—moduli—by the symbol $\hat{a} = (a_2, \dots, a_{3g-1})$, and the corresponding indices by the symbol $1, \dots, \widehat{3g-2}$.

3. We identify the space $T(S)$ with the domain D_{3g-3}. The domain $f_\mu(U_2)$, determined by a point $a = (a_2, a_3, \dots, a_{3g-1}) \in D_{3g-3}$, and the quasifuchsian group $\Gamma^\mu = f_\mu \Gamma f_\mu^{-1}$ will be denoted by $D_2(a)$ and Γ_a, respectively. By $w_a = f_a(z)$ we shall denote the mapping $f_\mu(z)$ for $\mu(z) = \mu_a(z) = \tau^{-1}(a) \in [\mu]$.

For the mappings $w_{\mu+\nu} = f_{\mu+\nu}(z)$, for $\|\nu\|_{L_\infty(U_2)} \le \varepsilon < \varepsilon_0$, $0 < \varepsilon_0 < 1$, the variational formula holds

$$w_{\mu+\nu} = w_\mu - \frac{w_\mu^2(w_\mu - 1)}{\pi} \iint_{w_\mu(U_2)} \frac{\theta(\zeta)\, d\sigma(\zeta)}{\zeta^2(\zeta-1)(\zeta - w_\mu)} + O(\varepsilon^2), \tag{1}$$

where the estimate of the remainder term is uniform for $|w_\mu| \le R < r_0(\varepsilon_0)$, $r_0(\varepsilon_0)$ is a completely determined function of ε_0 such that $\lim_{\varepsilon_0 \to 0} r_0(\varepsilon_0) = \infty$, and

$$\theta(w_\mu) = \left(\frac{\nu}{1 - \mu(\mu+\nu)} \frac{(w_\mu)_z}{(w_\mu)_z} \right) \circ (w_\mu)^{-1}. \tag{2}$$

It follows from this, in particular, that the mappings $f_a \circ f_{a_0}^{-1}(w)$ depend holomorphically on the moduli $a \in T(S)$ for fixed a_0 and w. Therefore the transformations $A_a = f_a \circ A \circ f_a^{-1}$, $A \in \Gamma$, of the group Γ_a (i.e. their coefficients) also depend holomorphically on the moduli a.

We shall also denote by $B(D_2(a), \Gamma_a)$ the complex Banach space of functions $\psi(w)$ holomorphic in the domain $D_2(a)$ such that $\psi(Aw){A'}^2(w) = \psi(w)$, $A \in \Gamma_a$, and $\psi(w) = O(|w|^{-4})$ near $w = \infty$, with norm $\|\psi\| = \sup_{w \in D_2(a)} \lambda_a^{-2}(w)|\psi(w)|$, where $\lambda_a(w)|dw|$ is the Poincaré metric of the domain $D_2(a)$.

We construct over $T(S)$ the fibered space

$$\widetilde{T}(S) = \{(a, w) \mid a \in T(S),\ w = f_a(z) \in D_2(a)\}.$$

The space $\widetilde{T}(S)$ has a natural complex structure. Its base is $T(S) = D_{3g-3}$, and the projection $(a, w) \mapsto a$ is defined so that the fiber over the point $a \in T(S)$ is the domain $D_2(a)$—the universal covering surface of

$$S_a = D_2(a)/\Gamma_a.$$

4. Put now

$$\psi_j(w, a) = \frac{1}{(j+1)!} \sum_{A \in \Gamma_a} \frac{d^{j+1}}{dz^{j+1}} \left[\frac{f_a^2(z)(f_a(z) - 1)}{Aw - f_a(z)} \right] \Bigg|_{z=0} \frac{{A'}^2(w)}{(Aw)^2(Aw - 1)}, \tag{3}$$

$$j = 1, 2, \dots, 3g-3.$$

Theorem. The functions $\psi^j_{i_j}(w, a)$, $j = 1, 2, \dots, 3g-3$, $i_j = 1, \dots, \widehat{3g-2}$, for $w = f_a(z)$ are holomorphic in w and a in $\widetilde{T}(S)$, and for each fixed $a \in T(S)$ form a basis in the space $B(D_2(a), \Gamma_a)$.

The holomorphy of the functions (3) in w and a is established by estimating the terms of these series. To prove their linear independence, one uses the variational formula (1), (2), as well as the connection between the moduli

$$\hat{a} = (a_2, a_3, \dots, \widehat{a_{3g-2}}) \in T(S)$$

and the global moduli of L. Ahlfors–L. Bers (see ([1], [3], [4])).

This theorem gives a simultaneous uniformization of all holomorphic quadratic differentials on the surfaces S_a. The formula

$$\psi(w, a) = \sum_{j=1}^{3g-3} \xi_j \psi_{i_j}(w, a), \qquad \xi_j = \text{const}, \quad i_j = 1, 2, \dots, \widehat{3g-2},$$

gives a "holomorphic pro-

"continuation" in $\tilde{T}(S)$ of each holomorphic quadratic differential on any surface S_a, $a_0 \in T(S)$.

The ratios $\psi_j(w, a)/\psi_i(w, a)$, $i \neq j$, are meromorphic functions on $\tilde{T}(S)$ and are rational for each $a \in T(S)$.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR
Novosibirsk

Received
31 III 1969

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Note: Figure translations are in progress. See original paper for figures.