ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Abstract

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ELLIPTIC BOUNDARY-VALUE PROBLEMS

FOR PLANE DOMAINS WITH CORNERS AND DISCONTINUITIES

REACHING THE BOUNDARY

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In the present note we state existence theorems for boundary-value problems satisfying the Lopatinskii condition in weighted spaces, analogous to those considered in (^{1}).

1°. Notation, formulation of the problem. Let Ω be a bounded plane domain with piecewise smooth boundary Γ, containing a finite number of nonzero angles, divided by a finite number of smooth curves γ_ν into parts Ω_μ. In Ω_μ let operators $L_\mu(x, y, \partial/\partial x, \partial/\partial y)$ be given, and on Γ, γ_ν, respectively, operators $B(x, y, \partial/\partial x, \partial/\partial y)$ and $C_\nu^\pm(x, y, \partial/\partial x, \partial/\partial y)$ in the form of matrices of dimensions respectively $N \times N$, $m \times N$, and $2m \times N$ (the number m will be defined below), whose elements $l_{ij}^{(\mu)}$, b_{ij}, and $c_{ij\nu}^\pm$ are homogeneous linear differential operators of orders respectively $s_i + t_j$, $\sigma_i + t_j$, $\sigma_i + t_j$, $\max_i s_i = 0$ (see (^{2})). The operators L_μ are uniformly elliptic and satisfy the additional condition on L (see (^{2})). Then the number m introduced above is equal to half the order of $\det L_\mu$. The coefficients entering into $l_{ij}^{(\mu)}$, b_{ij}, $c_{ij\nu}^\pm$ are assumed sufficiently smooth (at angular points and points of intersection of the curves this assumption can be weakened).

Consider the problem (see (^{3}))

$$L_\mu(x, y, \partial/\partial x, \partial/\partial y)U = f(x, y), \qquad (x, y) \in \Omega_\mu; \tag{1}$$

$$B(x, y, \partial/\partial x, \partial/\partial y)U = \varphi(x, y), \qquad (x, y) \in \Gamma; \tag{2}$$

$$C_\nu^+ U^+ - C_\nu^- U^- = \psi_\nu(x, y), \qquad (x, y) \in \gamma_\nu, \tag{3}$$

in the functional spaces

$$\|U_j\|^2_{\dot W^k_a(\Omega)} = \sum_{|\alpha|\le k} \iint_\Omega r^{a-2(k-|\alpha|)} |D^\alpha U_j| \, dx \, dy, \qquad j = 1, \dots, N,$$

$$\|\varphi\|_{\dot W^{k-1/2}_a(\partial\omega)} = \inf_\Pi \|\Pi\varphi\|_{\dot W^k_a(\omega)},$$

where $\Pi\varphi \in \dot W^k_a(\omega)$ is an extension of φ from the boundary $\partial\omega$ into the domain ω, and r is the product of the distances from the point (x, y) to all angular points and points of intersection of the pairs (Γ, γ_ν) and (γ_μ, γ_ν). The direct product of the spaces

$$\dot W^{k-s_1}_a \times \dots \times \dot W^{k-s_N}_a \times \dot W^{k-\sigma_1-1/2}_a \times \dots \times \dot W^{k-\sigma_m-1/2}_a$$

will be denoted by $\dot H^k_a$. Suppose that the operators $B, C^\pm_\nu$ jointly cover the operators L_μ in the sense of (^{3}).

2°. Problems with one angular point. First consider problem (1)–(2) in a domain not divided by the curves γ_ν, with boundary Γ containing only one angular point (for example, the origin), and pose the auxiliary problem

$$L(0, 0; \partial/\partial x, \partial/\partial y)U = f, \qquad (x, y) \in \Omega_0; \tag{4}$$

$$B(0, 0, \partial/\partial x, \partial/\partial y)U = \varphi, \qquad (x, y) \in \Gamma_0, \tag{5}$$

where Γ_0 consists of two rays issuing from the angular point tangent to Γ, and Ω_0 –

the interior of the angle between them. After the change of variables

$$x = e^{-\tau}\cos\omega, \qquad y = e^{-\tau}\sin\omega, \qquad U_j = e^{-t_j\tau} V_j(\omega, \tau), \qquad j = 1, \dots, N,$$

and the Fourier transform with respect to τ

$$\widetilde F(\lambda) = \int_{-\infty}^{+\infty} F(\tau) e^{-i\lambda\tau} \, d\tau$$

the auxiliary problem (4)–(5) is transformed into a boundary-value problem in the strip $0 < \omega < \omega_0$ for a system of ordinary differential equations with respect to the functions $V_j(\omega, \tau)$, $j = 1, \dots, N$,

$$\widetilde{L}(\omega, d/d\omega, i\lambda)\widetilde{V} = \widetilde{F}(\omega, \lambda); \tag{6}$$

$$\widetilde{B}(\omega, d/d\omega, i\lambda)\widetilde{V}\big|_{\omega=0;\,\omega_0} = \widetilde{\Phi}(\lambda), \tag{7}$$

where $\widetilde{L}$ and $\widetilde{B}$ are the Fourier transforms with respect to τ of matrices consisting, respectively, of the elements

$$e^{-(s_i+t_j)\tau} l_{ij}(0, 0, \partial/\partial x, \partial/\partial y), \qquad e^{-(\sigma_i+t_j)\tau} b_{ij}(0, 0, \partial/\partial x, \partial/\partial y)$$

and, by construction of the system, independent of τ. If problem (4)–(5) is properly elliptic and satisfies the Lopatinskii condition, then problem (6)–(7) satisfies conditions analogous to Conditions I and II for systems formulated in (4) for the case $s_i = 0$, $t_j = 2m$. Applying the method set forth in (4), one can show that for sufficiently large λ there exists a resolving operator $R(\lambda)$, acting from the intersection of the Sobolev spaces

$$W_2^{k-s_1} \times \cdots \times W_2^{k-s_N} \times W_2^{k-\sigma_1-1/2} \times \cdots \times W_2^{k-\sigma_m-1/2}$$

to the space

$$H^k = W_2^{k-t_1} \times \cdots \times W_2^{k-t_N},$$

which is a meromorphic function of λ and is such that $\widetilde{V} = R(\lambda)(\widetilde{F}, \widetilde{\Phi})$ is the solution of problem (6)–(7) satisfying the estimate

$$\int_0^{\omega_0} \sum_{q=0}^{k+t_j} \left[1 + |\lambda|^{2(k+t_j-q)}\right] \left|\frac{d^q \widetilde{V}_j}{d\omega^q}\right|^2 d\omega \leq C \left\{ \int_0^{\omega_0} \sum_{i=1}^{N} \left(\left|\frac{d^{\,k-s_i} \widetilde{F}_i}{d\omega^{\,k-s_i}}\right|^2 + |\lambda|^{k-s_i} \widetilde{F}_i|^2 \right) d\omega + \right.$$

$$\left. + \sum_{i=1}^{m} |\lambda|^{2(k-\sigma_i)-1} \left(|\widetilde{\Phi}_i|_{\omega=0}|^2 + |\widetilde{\Phi}_i|_{i=\omega_0}|^2 \right) \right\}. \tag{8}$$

Theorem 1. Suppose that the resolving operator $R(\lambda)$ for problem (4)–(5) has no poles on the line $\operatorname{Im}\lambda = h$. Then for $(f, \varphi) \in \dot{H}^k_{2\,(k-h-1)}$ there exists a unique solution of problem (4)–(5), satisfying the estimate

$$\|U_j\|_{\dot{W}_2^{k+t_j}{}_{(k-h-)}} \leq C\|f, \varphi\|_{\dot{H}^k_{2\,(k-h-1)}}. \tag{9}$$

Theorem 2. If $R(\lambda)$ for the auxiliary problem (4)–(5) has no poles on the line $\operatorname{Im}\lambda = h$, then for $(f, \varphi) \in \dot{H}^k_{2\,(k-h-1)}$, under the fulfillment of a finite number of conditions of the type of functionals in $\dot{H}^{k}_{2\,(k-h-1)}$, $F_p(f, \varphi) = 0$, there exists a solution of problem (1)–(2)

$$U_j \in \dot{W}_2^{k+t_j}{}_{(k-h-1)}, \qquad j = 1, \ldots, N.$$

If this solution is unique, then the estimate (9) is valid for it.

3°. Adjoint problems. We now consider problem (1)–(3) with one line of discontinuity γ, intersecting Γ at nonzero angles at the origin and at the point Q. Rotate the coordinate axes so that the x-axis becomes tangent to Γ, and the y-axis coincides with the inward normal. Then the equations of the curves Γ, γ in a neighborhood of the origin can be represented in the form $y = y_\Gamma(x)$, $y = y_\gamma(x)$. Since the angle between Γ and γ is nonzero, the transformation $x_1 = y - y_\Gamma$, $y_1 = y - y_\gamma$ is a smooth nondegenerate one, carrying problem (1)–(3) into the corresponding problem for the half-plane. Putting further, for $y < 0$, $z = -y$, $U_j(x, -z) = V_j(x, z)$, we see that,

that problem (1)–(3) is equivalent to the "doubled" boundary-value problem (1)–(2) with one straight angle, i.e., to a problem with twice as many equations and boundary conditions; moreover, proper ellipticity and the covering condition are preserved.

Consider the auxiliary problem

$$L(0, 0, \partial/\partial x, \partial/\partial y)U = f, \qquad (x, y) \in \Omega_0; \tag{10}$$

$$B(0, 0, \partial/\partial x, \partial/\partial y)U = \varphi, \qquad (x, y) \in \Gamma_0; \tag{11}$$

$$C^+(0, 0, \partial/\partial x, \partial/\partial y)U^+ - C^-(0, 0, \partial/\partial x, \partial/\partial y)U^- = \psi, \qquad (x, y) \in \gamma_0, \tag{12}$$

where Ω_0, Γ_0 are the same as in problem (4)–(5), and γ_0 is the tangent to γ. Passing to the "doubled" problem, we construct $R(\lambda)$.

Theorem 3. If $R(\lambda)$ of the "doubled" problem (10)–(12) has no poles on the line $\operatorname{Im} \lambda = h$, then for $(f, \varphi, \psi) \in \dot{H}^{k}_{2(k-h-1)}$ there exists a unique solution of problem (10)–(12) satisfying the estimate

$$\|U_j\|_{\dot{W}^{k+t_j}_{2(k-h-1)}} \le C \|f, \varphi, \psi\|_{\dot{H}^{k}_{2(k-h-1)}}. \tag{13}$$

If the domain Ω is divided by two intersecting curves γ_1 and γ_2, which do not have a common tangent at the point of intersection, then, after performing the transformations $x_1 = y - y_{\gamma_1}$, $y_1 = y - y_{\gamma_2}$, by analogous reasoning we obtain a “quadrupled” boundary-value problem with one corner. Consider the auxiliary problem

$$L(0,0,\partial/\partial x,\partial/\partial y)U=f,\qquad (x,y)\in\Omega_0; \tag{14}$$

$$C_\nu^+(0,0,\partial/\partial x,\partial/\partial y)U^+-C_\nu^-(0,0,\partial/\partial x,\partial/\partial y)U^-=\varphi_\nu,\qquad (x,y)\in\gamma_{\nu 0},\quad \nu=1,2. \tag{15}$$

Theorem 4. If $R(\lambda)$ of the “quadrupled” problem (14)—(15) has no poles on the line $\operatorname{Im}\lambda=h$, then for $(f,\varphi)\in\dot{H}^k_{2(k-h-1)}$ there exists a unique solution of problem (14)—(15) satisfying the estimate

$$\|U_j\|_{\dot{W}^{k+t_j}_{2(k-h-1)}}\le C\|f,\varphi\|_{\dot{H}^k_{2(k-h-1)}}.$$

Theorems 1, 3, and 4 make it possible to construct a regularizer for problem (1) —(3) (see ([1],[4],[5])); therefore the following is true.

Theorem 5. If R_ν of the auxiliary problems (4)—(5) for each corner point, R_μ of the “doubled” auxiliary problems for each point of intersection of Γ,γ, and $R_{\mu\nu}$ of the “quadrupled” auxiliary problems for each point of intersection of the curves (γ_μ,γ_ν) have no poles on one and the same line $\operatorname{Im}\lambda=h$, then for $(f,\varphi,\psi)\in\dot{H}^k_{2(k-h-1)}$, upon satisfaction of a finite number of conditions of the type of functionals in $\dot{H}^k_{2(k-h-1)}$, there exists a solution of problem (1)—(3), $U_j\in\dot{W}^{k+t_j}_{2(k-h-1)}$, $j=1,\dots,N$. If it is unique, then an estimate of type (13) is valid.

Remark. The curves γ_ν may close up without reaching the boundary. If they do not intersect, then normal solvability holds (see ([3])).

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Note: Figure translations are in progress. See original paper for figures.