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Abstract

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ON A SYSTEM OF BOUNDARY-LAYER EQUATIONS FOR AN AXISYMMETRIC STATIONARY FLOW OF A COMPRESSIBLE FLUID

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The question of the existence, uniqueness, and stability of solutions of the system of boundary-layer equations for stationary and nonstationary flows of an incompressible fluid has been studied in the works of O. A. Oleinik (see ([1])). In ([2]) the problem of continuation of the boundary layer for a stationary flow of a compressible fluid is considered under the assumption that the Prandtl number σ is equal to one.

In the present note we give results concerning the existence and uniqueness of the solution of the system of boundary-layer equations for a stationary axisymmetric flow of a compressible fluid for $\sigma = 1$.

For $\sigma = 1$ there exists a quadratic relation between the enthalpy and the longitudinal component of velocity ([3]). Therefore we consider the dynamical system of equations

$$\rho u \frac{\partial u}{\partial x} + \rho \vartheta \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right),$$
$$\frac{\partial(\rho r_0 u)}{\partial x} + \frac{\partial(\rho r_0 \vartheta)}{\partial y} = 0 \qquad (1)$$

in the domain $D_x\{0 \le x \le X,\ 0 \le y < \infty\}$ under the conditions

$$u\big|_{x=0} = 0, \qquad u\big|_{y=0} = 0, \qquad \vartheta\big|_{y=0} = \vartheta_0(x), \qquad \lim_{y\to\infty} u(x,y) = u_l(x). \qquad (2)$$

The function $r_0(x)$ determines the surface of the body being flowed around, $r_0(0) = 0$, $r_0'(0) \neq 0$, $r_0(x) > 0$ for $x > 0$; $u_l(x)$ is the longitudinal component of the velocity at the outer boundary of the boundary layer, $u_l(0) = 0$, $u_l(x) > 0$ for $x > 0$, $u_l'(x) > 0$ for $x \geq 0$. In the case under consideration we have

$$\mu = \mu_0 f\left(1 - \frac{u^2}{2h_0}\right), \qquad \rho = \rho_0\left(1 - \frac{u_l^2}{2h_0}\right)^{k/(k-1)}\left(1 - \frac{u^2}{2h_0}\right)^{-1}, \qquad (3)$$

$$p = p_0\left(1 - \frac{u_l^2}{2h_0}\right)^{k/(k-1)},$$

where f is a certain prescribed positive function; $k, h_0, \mu_0, \rho_0, p_0$ are certain positive constants (see ([3])).

We shall study problem (1), (2) by the methods developed in ([1]). If we introduce new independent variables

$$\xi = x, \qquad \eta = u(x,y)/u_l(x), \qquad (4)$$

then for the function $W = \mu u_y/u_l$ we obtain the equation

$$W^2 W_{\eta\eta} - \eta A W_\xi + B W_\eta + C W = 0 \qquad (5)$$

in the domain $G_x\{0 \leq x \leq X,\ 0 \leq \eta < 1\}$ with the conditions

$$W\big|_{\eta=1} = 0, \qquad \left(W W_\eta - D W + E\right)\big|_{\eta=0} = 0, \qquad (6)$$

where $A(\xi,\eta) = a(\xi,\eta) u_l$,

$$a(\xi,\eta) \equiv \mu\rho = \mu_0\rho_0(1-\alpha_l^2)^{k/(k-1)}\frac{f(1-\alpha_l^2\eta^2)}{1-\alpha_l^2\eta^2},$$

$$B(\xi,\eta) = a\frac{\eta^2-1}{1-\alpha_l^2}u_l', \qquad C(\xi,\eta) = \eta A\frac{r_0'}{r_0} + \eta\frac{\partial A}{\partial \xi} - \frac{\partial B}{\partial \eta}, \qquad \alpha_l = \frac{u_l}{\sqrt{2h_0}},$$

$$D = \rho_0(1-\alpha_l^2)^{k/(k-1)} v_0(\xi), \qquad E = \rho_0\mu_0 f(1)(1-\alpha_l^2)^{1/(k-1)} u_l'.$$

The boundary condition (6) for $\eta = 0$ means that the first equation of the system (1) is satisfied for $y = 0$, while the condition for $\eta = 1$ ensures fulfillment of the condition $u \to u_l$ as $y \to \infty$.

Equation (5), under the conditions of the problem on continuation of the boundary layer, was considered in (2). It is proved there that equation (5) has a solution in the class of functions W possessing bounded first derivatives in G_X. We note that problem (5), (6) has no solution in the indicated class for arbitrarily smooth data. Therefore, below we shall establish existence and uniqueness theorems for the solution of problem (5), (6) in another class of functions, namely in the class of functions possessing certain singularities at $\eta = 1$ (see (1)).

Let $\varphi^k(\eta) \equiv \varphi(kh, \eta)$. Following O. A. Oleinik, we replace equation (5) with conditions (6) by the system of ordinary differential equations

$$(W^k)^2 W^k_{\eta\eta} - (\eta A^k + \mu_k h)(W^k - W^{k-1})/h + B^k W^k_\eta + C^k W^k = 0, \qquad (7)$$

$$0 \le \eta < 1, \qquad k = 0, 1, 2, \dots, [X/h]$$

with the conditions

$$W^{(k)}(1) = 0, \qquad (W^k W^k_\eta - D^k W^k + E^k)\big|_{\eta=0} = 0, \qquad (8)$$

where μ_k is a certain positive constant for $k \ge 1$, determined by the data of the problem, and $\mu_0 = 0$.

We shall assume that the functions f, u_l, r_0 are twice continuously differentiable, and that $v_0(x)$ has a bounded derivative; moreover, $|u_l''| \le N_1 x$, $v_0(x) \le N_2 x$, $N_i = \text{const} > 0$.

Lemma 1. The problem (7), (8) has a solution $W^k(\eta)$, continuous for $0 \le \eta \le 1$ and possessing, for $0 \le \eta < 1$, derivatives up to and including the third order. For this solution $W^k(\eta)$ the estimate holds

$$M_1(1-\eta)\sigma \le W^k(\eta) \le M_2(1-\eta)\sigma, \qquad \sigma = \sqrt{-\ln \mu(1-\eta)} \qquad (9)$$

for $kh \le X$, where $M_1, M_2, 1 > \mu$ are positive constants independent of h.

Lemma 2. For $kh \le X_1$, where $X_1 > 0$ depends on f, u_l, r_0, v_0, the following estimates are valid for the solution $W^k(\eta)$ of problem (7), (8):

$$-M_3\sigma \le W^k_\eta \le M_4\sigma,$$

$$\frac{1}{h}\left|W^k - W^{k-1}\right| \le M_5(1-\eta)\sigma, \qquad (10)$$

$$W^k W^k_{\eta\eta} \le -M_6, \qquad \left|W^k W^k_{\eta\eta\eta}\right| \le M_7,$$

where σ is the same as in Lemma 1, and M_i are positive constants independent of h.

Theorem 1. Suppose that the assumptions stated above concerning the functions f, u_l, r_0, v_0 are fulfilled. Then in the domain $G_{X_1}\{0 \leq \xi \leq X_1,\ 0 \leq \eta < 1\}$ there exists a unique solution of problem (5), (6) having the following properties: $W(\xi, \eta)$ is continuous in $\overline{G}_{X_1}$ and $\hat{M}_1(1-\eta)\sigma \leq W \leq \hat{M}_2(1-\eta)\sigma$; W_η is continuous in η for $\eta < 1$, $-M_3\sigma \leq W_\eta \leq -M_4\sigma$ in G_{X_1}; $|W_\xi| \leq M_5(1-\eta)\sigma$; $WW_{\eta\eta} < 0$ and $WW_{\eta\eta\eta}$ is bounded in $\widehat{G}_{X_1}$. In any closed domain lying strictly inside G_{X_1}, W and its derivatives, in-

entering equation (5) satisfy the Hölder condition. The function W satisfies equation (5) at all interior points of G_{X_1}, and for $0 \leq \xi \leq X_1$ the conditions (6) are fulfilled.

The uniqueness of the solution W of problem (5), (6) follows from the maximum principle applied to the equation for the difference of two solutions. The functions $W^h(\eta)$, which are solutions of equations (7) with conditions (8), form, by virtue of Lemmas 1 and 2, a compact family in the sense of uniform convergence. By virtue of the estimates (9), (10), the limiting (as $h \to 0$) function W is continuous in G_{X_1} and has the properties indicated in Theorem 1.

Since at the interior points of G_{X_1} equation (5) is parabolic, the derivatives $W_\xi, W_\eta, W_{\eta\eta}$ satisfy the Hölder condition in any subdomain G_{X_1}, and inside G_{X_1}, W satisfies equation (5). By virtue of the boundedness of $W^h_{\eta\eta}$ for $\eta < 1$, the derivative W_η satisfies the Lipschitz condition with respect to η for $\eta < 1$, and condition (6) is fulfilled for $\eta = 0$. As a consequence of Theorem 1 we obtain the following assertion.

Theorem 2. *For $x \leq X_1$ in the domain D_{X_1} there exists a unique solution of problem (1), (2) possessing the following properties: u/u_1, u_y/u_1 are bounded and continuous in D_{X_1}, $u > 0$ for $y > 0$ and $x > 0$, $u(x,y) \to u_1(x)$ as $y \to \infty$; $u_y/u_1 > 0$ for $y \geq 0$, $u_y/u_1 \to 0$ as $y \to \infty$; u_x, v_y, u_{yy} are bounded and continuous with respect to y in D_{X_1} and continuous with respect to x and y at the interior points of D_{X_1}; $v(x,y)$ is continuous with respect to y in D_{X_1}, with respect to x and y at the interior points of D_{X_1}, and is bounded for bounded y; the derivative u_{yyy} is bounded in D_{X_1}, u_{xy} is bounded for bounded y; u_{xy} and u_{yyy} are continuous at the interior points of D_{X_1}; u_{yy}/u_y is continuous with respect to y in D_{X_1}. The functions u, v satisfy inside D_{X_1} equations (1) and $u|_{x=0} = u|_{y=0} = 0$; $v|_{y=0} = v_0(x)$.*

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CITED LITERATURE

1. O. A. Oleinik, *Uspekhi Mat. Nauk*, **23**, no. 3, 3 (1968).

2. T. D. Dzhuraev, *Izv. AN UzSSR*, Ser. Phys.-Math. Sci., No. 6 (1968).

3. L. G. Loitsyanskii, *Laminar Boundary Layer.* Moscow, 1962.