# CONSISTENCY OF A CERTAIN FRAGMENT OF QUINE’ S SYSTEM NF

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## Abstract

## Full Text

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**MATHEMATICS**

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# CONSISTENCY OF A CERTAIN FRAGMENT OF QUINE' S SYSTEM NF

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Let us denote by $\mathrm{NF}_n$ the fragment of NF (New Foundations) containing those axioms of NF (see ([1])) whose stratifiability (stratification) can be established using numbers not exceeding $n$, i.e. the numbers $1, 2, 3, \ldots, n$. For example, to check the stratification of the formula

$$\forall z \exists y \forall x (x \varepsilon y \equiv \forall t (t \varepsilon x \to t \varepsilon z))$$

the numbers $1, 2, 3$ are required (the variables $t, x, z, y$ are assigned the numbers $1, 2, 2, 3$, respectively). The axiom of NF mentioned will therefore be an axiom of $\mathrm{NF}_3$.

The consistency of $\mathrm{NF}_3$ is proved. We note without proof that from the consistency of $\mathrm{NF}_5$ the consistency of NF follows.

Let $T_n$ denote type theory without the axiom of infinity, having $n$ type variables, i.e. containing $n$ alphabets of variables. In addition to the logical axioms, the axioms of $T_n$ are the formulas

$$\forall x^i (x^i \varepsilon y^{i+1}) \equiv (x^i \varepsilon z^{i+1}) \to y^{i+1} = z^{i+1},$$

$$\exists y^{i+1} \forall x^i (x^i \varepsilon y^{i+1} \equiv A(x^i)) \quad (i = 1, 2, \ldots, n-1),$$

where $x^i, y^{i+1}, z^{i+1}$ are variables of the $i$-th and $(i+1)$-st alphabets (types), respectively, and the variable $y^{i+1}$ does not occur freely in the formula $A(x^i)$. We note that the special (nonlogical) axioms of $\mathrm{NF}_n$ are the formulas obtained from the stated axioms of $T_n$ by erasing the upper, i.e. type, index.

By a result of Specker ([2]), in order to prove the consistency of $\mathrm{NF}_3$ it suffices to find a model $M = (X_1, X_2, X_3, \in, =)$ (where $X_2 \subseteq \{A : A \subseteq X_1\}$ and $X_3 \subseteq \{A : A \subseteq X_2\}$) for $T_3$ admitting an $\varepsilon$-isomorphism, i.e. admitting the existence of such one-to-one functions $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_3$ that

$$a \in b \equiv f_1(a) \in f_2(b)$$

for any $a \in X_1,\ b \in X_2$. The existence of an $\varepsilon$-isomorphism is equivalent to the existence of an isomorphism $\varphi$ with respect to inclusion ($\subseteq$) between the domains $X_2$ and $X_3$. If one defines $f_1$ by the equivalence

$$b = f_1(a) \equiv \{b\} = \varphi(\{a\})$$

for any $a \in X_1$ and $b \in X_2$ (where $\{a\}$ and $\{b\}$ denote one-element sets containing $a$ and $b$, respectively) and sets $f_2 = \varphi$, then the system $f_1, f_2$ is an $\varepsilon$-isomorphism.

The following lemma gives some sufficient conditions for two families of subsets $\mathfrak{A}$ and $\mathfrak{B}$ of sets $X$ and $Y$ to be isomorphic with respect to inclusion.

**Lemma.** *Let two families of subsets $\mathfrak{A}$ and $\mathfrak{B}$ of sets $X$ and $Y$ satisfy the following conditions:*

1°. $|\mathfrak{A}| = |\mathfrak{B}| = \aleph_0$, *where $|\mathfrak{A}|$ and $|\mathfrak{B}|$ are the cardinalities of $\mathfrak{A}$ and $\mathfrak{B}$, respectively.*

2°. $\mathfrak{A}$ and $\mathfrak{B}$ are closed under the operations of intersection and complement to $X$ and $Y$, respectively.

3°. $\mathfrak{A}$ and $\mathfrak{B}$ contain all one-element subsets of $X$ and $Y$, respectively.

4°. For every infinite $A \in \mathfrak{A}$ (or $B \in \mathfrak{B}$) there exists $A_1 \in \mathfrak{A}$ ($B_1 \in \mathfrak{B}$) such that $A_1 \subseteq A$ and $|A_1| = |A - A_1|$ ($B_1 \subseteq B$ and $|B_1| = |B - B_1|$).

Then the families $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic with respect to inclusion.

**Proof.** Construct the following enumerations without repetitions

$$A_1, A_2, A_3, \dots, A_k, \dots, \qquad B_1, B_2, B_3, \dots, B_k, \dots$$

of the families $\mathfrak{A}$ and $\mathfrak{B}$, respectively, so that the equality

$$\left|A_1^{\sigma_1} \cap \dots \cap A_k^{\sigma_k}\right| = \left|B_1^{\sigma_1} \cap \dots \cap B_k^{\sigma_k}\right| \tag{1}$$

holds for every $k$ and every set $\sigma_1, \dots, \sigma_k$ of zeros and ones, where

$$A^\sigma = \begin{cases} A, & \text{if } \sigma = 0, \\ -A, & \text{if } \sigma = 1. \end{cases}$$

For this purpose fix some enumerations of the families $\mathfrak{A}$ and $\mathfrak{B}$. In view of condition 2°, $X \in \mathfrak{A}$ and $Y \in \mathfrak{B}$. Put $A_1 = X$ and $B_1 = Y$. Obviously, equality (1) holds for $k = 1$. Suppose that $A_1, \ldots, A_n, B_1, \ldots, B_n$ have already been defined, contain no repetitions, and satisfy equality (1).

Let $n$ be odd. Put $A_{n+1}$ equal to the first element, in the fixed enumeration of the family $\mathfrak{A}$, that is different from $A_1, \ldots, A_n$. Denote by $\mathfrak{D}_n$ the family consisting of sets of the form $A_1^{\sigma_1} \cap \ldots \cap A_n^{\sigma_n}$, where $\sigma_1, \ldots, \sigma_n$ runs through all possible sets of zeros and ones. Denote by $\mathfrak{D}'_n$ the family of sets of the form $B_1^{\sigma_1} \cap \ldots \cap B_n^{\sigma_n}$. An arbitrary element of $\mathfrak{D}_n$ is denoted by $\Delta$, and the corresponding element of $\mathfrak{D}'_n$, i.e. the element with the same set $\sigma_1, \ldots, \sigma_n$, is denoted by $\Delta'$. For an arbitrary element $\delta$ of the family $\mathfrak{D}_n$ or $\mathfrak{D}'_n$ and an arbitrary subset $C$ of the set $X$ or $Y$, define

$$\tau_\delta^C = |C \cap \delta|, \qquad \chi_\delta^C = |-C \cap \delta|,$$

where $-C$ is the complement of the set $C$. Consider the numbers $\tau_\Delta^{A_{n+1}}$ and $\chi_\Delta^{A_{n+1}}$ ($\Delta \in \mathfrak{D}_n$). The numbers $\tau_\Delta^{A_{n+1}}$ and $\chi_\Delta^{A_{n+1}}$ characterize the "behavior" of the set $A_{n+1}$ with respect to the "pieces" $\Delta$ from $\mathfrak{D}_n$. Define $B_{n+1}$ as that element of $\mathfrak{B}$ whose "behavior" with respect to $\mathfrak{D}'_n$ coincides with the "behavior" of $A_{n+1}$ with respect to $\mathfrak{D}_n$. More precisely, $B_{n+1}$ is equal to the first element, in the fixed enumeration of the family $\mathfrak{B}$, such that

$$\tau_{\Delta'}^{B_{n+1}} = \tau_\Delta^{A_{n+1}}, \qquad \chi_{\Delta'}^{B_{n+1}} = \chi_\Delta^{A_{n+1}}$$

for every $\Delta \in \mathfrak{D}_n$.

Let us prove the existence of such a $B_{n+1}$. We have:

$$\tau_\Delta^{A_{n+1}} + \chi_\Delta^{A_{n+1}} = |\Delta|$$

for every $\Delta \in \mathfrak{D}_n$, and $|\Delta| = |\Delta'|$. By 1° and 3°, the numbers $\tau_\Delta^{A_{n+1}}$ and $\chi_\Delta^{A_{n+1}}$ do not exceed $\aleph_0$. If both of the numbers mentioned are equal to $\aleph_0$, then the existence of $\Delta' \cap B_{n+1}$ follows from condition 4°. If one of these numbers is less than $\aleph_0$, then the existence of $\Delta' \cap B_{n+1}$ follows from 2° and 3°. Obviously, equality (1) holds for $k = n + 1$. Let us prove that $B_{n+1} \neq B_i$ for every $i = 1, \ldots, n$. If $B_{n+1} = B_i$ for some $i \leq n$, then

$$\tau_\Delta^{A_{n+1}} = \tau_{\Delta'}^{B_{n+1}} = \tau_{\Delta'}^{B_i} = \tau_\Delta^{A_i}, \tag{2}$$

$$\chi_\Delta^{A_{n+1}} = \chi_{\Delta'}^{B_{n+1}} = \chi_{\Delta'}^{B_i} = \chi_\Delta^{B_i}. \tag{3}$$

From these equalities it follows that $A_{n+1} = A_i$. Indeed, let $a \in A_i$. Then there is a $\Delta \in \mathfrak{D}_n$ such that

$$a \in \Delta = A_1^{\sigma_1} \cap \cdots \cap A_i^{\sigma_i} \cap \cdots \cap A_n^{\sigma_n}.$$

Since $a \in A_i$, we have $\sigma_i = 0$, i.e. $\Delta \subseteq A_i$, or $\chi_\Delta^{A_i} = 0$. But $\chi_\Delta^{A_i} = \chi_\Delta^{A_{n+1}}$ (in view of (3)). Therefore $\Delta \subseteq A_{n+1}$, i.e. $a \in A_{n+1}$. Let $a \in -A_i$. Then there is a $\Delta \in \mathfrak{D}_n$ such that

$$a \in \Delta = A_1^{\sigma_1} \cap \dots \cap A_i^{\sigma_i} \cap \dots \cap A_n^{\sigma_n}.$$

Next we derive: $\sigma_i = 1$, $\Delta \subseteq -A_i$, $\tau_\Delta^{A_i} = 0$, and $\Delta \subseteq -A_{n+1}$, and, finally, $a \in -A_{n+1}$. However, by virtue of its choice, $A_{n+1}$ is distinct from $A_1, \dots, A_n$.

Let $n$ be even. Put: $B_{n+1}$ equal to the first, in the fixed enumeration of the family $\mathfrak{B}$, element distinct from $B_1, \dots, B_n$. $A_{n+1}$ is defined in the same way as $B_{n+1}$ was defined in the case of odd $n$.

The sequences constructed are, obviously, enumerations, and moreover without repetitions, of the families $\mathfrak{A}$ and $\mathfrak{B}$, satisfying equality (1). From condition (1) it follows that

$$A_i \subseteq A_j \equiv B_i \subseteq B_j$$

for any $i$ and $j$. The lemma is proved.

It remains to find such a model $M = (X_1, X_2, X_3, \in, =)$ for $T_3$ that the families of sets $X_2$ and $X_3$ satisfy the condition of the lemma proved. Then the domains $X_2$ and $X_3$ will be isomorphic with respect to inclusion, and the model $M$ admits an $\varepsilon$-isomorphism.

Consider the theory denoted by $T_5^\infty$, obtained from the theory of types $T_5$ by extending both the language $T_5$ by adding an infinite list of constants $c_1, c_2, c_3, \dots, c_m, \dots$, belonging to type 1, and the axioms of $T_5$ by introducing the following axioms:

(I) $c_i \neq c_j$ for $i \neq j$.

(II) $\forall x^{i+1}\,\mathrm{Fin}(x^{i+1})$ $(i = 1, 2, 3)$, where $\mathrm{Fin}(x^{i+1})$ is the formula of $T_5$ expressing the predicate "$x^{i+1}$ is finite."

Obviously, the theory $T_5^\infty$ is consistent. From the consistency of the theory $T_5^\infty$ there follows the existence of a countable model for it. The part of this model formed by the domains $X_1, X_2, X_3$ is a model for $T_3$. Obviously, conditions $1^0, 2^0, 3^0$ of the lemma are fulfilled for the domains $X_2$ and $X_3$. It remains to verify condition $4^0$. Denote by $\mathrm{Zr}(x^{i+1})$ $(i = 1, 2)$ the formula of the theory $T_5$ expressing the following judgment: "either the set $x^{i+1}$, or $x^{i+1}$ with one element removed, decomposes into two equipotent disjoint sets." In the theory $T_5$ the formula is provable

$$\forall x^{i+1}(\mathrm{Fin}(x^{i+1}) \to \mathrm{Zr}(x^{i+1})) \qquad (i = 1, 2).$$

In the theory $T_5^\infty$, in view of axiom (II), the formula is provable

$$\forall x^{i+1}\,\mathrm{Zr}(x^{i+1}) \qquad (i = 1, 2). \tag{4}$$

In the model for $T_5^\infty$ formula (4) is satisfied, and therefore for every $A \in X_{i+1}$ $(i = 1, 2)$ either $A$, or $A$ with one element removed, decomposes into two equipotent disjoint sets. Hence the validity of condition $4^0$ of the lemma follows, if $X_2$ is taken as $\mathfrak{A}$, and $X_3$ as $\mathfrak{B}$.

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[1] W. V. O. Quine, *Am. Math. Monthly*, **44**, 70 (1937). [2] E. Specker, Typical Ambiguity, Proc. Intern. Congr., 1960, p. 116.

*Note: Figure translations are in progress. See original paper for figures.*