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ON A SINGULAR NICOLETTI PROBLEM

MATHEMATICS

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Abstract

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MATHEMATICS

I. T. Kiguradze

ON A SINGULAR NICOLETTI PROBLEM

(Presented by Academician I. N. Vekua on 11 XI 1968)

In the present note we consider the boundary-value problem

$$dx_i/dt = f_i(t, x_1, \dots, x_n) \quad (i = 1, 2, \dots, n); \quad (1)$$

$$x_i(t_i) = 0 \quad (i = 1, 2, \dots, n), \quad (2)$$

where $f_i(t, x_1, \dots, x_n)$ ($i = 1, 2, \dots, n$) are real functions defined in the domain $R_{a-b}(a \leq t \leq b, -\infty < x_1, \dots, x_n < +\infty)$, $-\infty < a < b < +\infty$, and $t_i \in [a, b]$ ($i = 1, 2, \dots, n$). By a solution of this problem we mean absolutely continuous functions $x_1(t), \dots, x_n(t)$ on $[a, b]$ satisfying the system (1) for almost all $t \in [a, b]$ and the boundary conditions (2).

Problem (1)–(2) was posed by O. Nicoletti ⁽⁴⁾. Various sufficient conditions for the existence and uniqueness of its solution were given in ^(1–7). In contrast to the cases considered by other authors, in the present note it is assumed that the functions $f_i(t, x_1, \dots, x_n)$ ($i = 1, 2, \dots, n$), having singularities at $t = t_i$ ($i = 1, 2, \dots, n$), in general are not summable with respect to t on the interval $[a, b]$.

Below we adopt the following notation:

The notation $\varphi(t, x_1, \dots, x_m) \in K(a, b)$, where m is an arbitrary natural number, means that the function $\varphi(t, x_1, \dots, x_m)$, for every $r \in (0, +\infty)$, satisfies the Carathéodory conditions in the domain $a \leq t \leq b$, $-r \leq x_1, \dots, x_m \leq r$, i.e., $\varphi(t, x_1, \dots, x_m)$ is measurable in t on the interval $[a, b]$ for all $x_i \in [-r, r]$ ($i = 1, 2, \dots, m$), is continuous in x_1, \dots, x_m in the domain $-r \leq x_1, \dots, x_m \leq r$ for almost all $t \in [a, b]$, and

$$\sup_{|x_i| \leq r \ (i=1,2,\dots,m)} |\varphi(t, x_1, \dots, x_m)| \in L(a, b).$$

By $\overline{K}(a, b; t_0)$, where $t_0 \in [a, b]$, is denoted the set of all functions belonging to the set $K(\alpha, \beta)$ for all α and β satisfying $a \leq \alpha < \beta \leq b$, $t_0 \in [\alpha, \beta]$.

$$l_k = (k-1)^{1/k} \left(\frac{k}{\pi} \sin \frac{\pi}{k} \right)^{-1}.$$

Theorem 1. *Let*

$$f_i(t, x_1, \dots, x_n) \in \overline{K}(a, b; t_i) \quad (i = 1, 2, \dots, n) \quad (3)$$

and let the inequalities

$$f_i(t, x_1, \dots, x_n) \operatorname{sign}[(t - t_i)x_i] \leq g_i(t, |x_1|, \dots, |x_n|) \quad (i = 1, 2, \dots, n),$$

hold in the domain R_{ab} , where the functions $g_i(t, x_1, \dots, x_n) \in K(a, b)$ ($i = 1, 2, \dots, n$) are nonnegative and nondecreasing in each x_j , $1 \leq j \leq n$. Moreover, suppose that there exists a positive number M such that

$$\rho_i(t) \leq M \quad \text{for } a \leq t \leq b \quad (i = 1, 2, \dots, n)$$

whatever absolutely continuous functions $\rho_i(t)$ ($i = 1, 2, \dots, n$) on the interval $[a, b]$ may be that satisfy on this interval the conditions

$$\begin{aligned} \rho_i(t_i) = 0, \quad 0 \leq \rho'_i(t) \operatorname{sign}(t - t_i) \leq g_i(t, \rho_1(t), \dots, \rho_n(t)) \\ (i = 1, 2, \dots, n). \end{aligned} \quad (4)$$

Then problem (1)–(2) has at least one solution.

Corollary 1. Suppose that conditions (3) are satisfied and, in the domain R_{ab} , the inequalities

$$\begin{aligned} f_i(t, x_1, \dots, x_n) \operatorname{sign}[(t - t_i)x_i] \leq \\ \leq h_i(t, |x_1|, \dots, |x_{i-1}|)\omega_i(|x_i|) \quad (i = 1, 2, \dots, n), \end{aligned}$$

hold, where $h_1(t, x_0) \equiv h_1(t) \in L(a, b)$, $h_i(t, x_1, \dots, x_{i-1}) \in K(a, b)$ ($i = 2, \dots, n$), and $\omega_i(x)$ ($i = 1, 2, \dots, n$) are continuous, positive, nondecreasing functions on the interval $[0, +\infty)$ satisfying the conditions

$$\int_0^{+\infty} \frac{dt}{\omega_i(t)} = +\infty \quad (i = 1, 2, \dots, n).$$

Then problem (1)–(2) has at least one solution.

Corollary 2. Suppose that conditions (3) are satisfied and, in the domain R_{ab} , the inequalities

$$f_i(t, x_1, \dots, x_n) \operatorname{sign}[(t - t_i)x_i] \leq$$

$$\leq g_i \left(t, \sum_{j=1}^n |x_j| \right) |x_i| + h_i \left(t, \sum_{j=1}^n |x_j| \right) \quad (i = 1, 2, \dots, n),$$

hold; the functions $g_i(t, x) \in K(a, b)$ and $h_i(t, x) \in K(a, b)$ ($i = 1, 2, \dots, n$) are nonnegative and nondecreasing in x , and

$$\max_{a \leq t \leq b} \sum_{i=1}^n \left| \int_{t_i}^t \left(h_i(\tau, x) \exp \left\{ \left| \int_{\tau}^t g_i(s, x) ds \right\} \right) d\tau \right| < x \quad \text{for } x > M,$$

where M is some positive constant. Then problem (1)–(2) has at least one solution.

Corollary 3. Suppose that conditions (3) are satisfied and, in the domain R_{ab} , the inequalities

$$f_i(t, x_1, \dots, x_n) \operatorname{sign}[(t - t_i)x_i] \leq g_i \left(t, \sum_{j=1}^n |x_j|^k \right) \quad (i = 1, 2, \dots, n), \quad (5)$$

hold, $1 < k < +\infty$, and the functions $g_i(t, x) \in K(a, b)$ ($i = 1, 2, \dots, n$) are nonnegative, nondecreasing in x , and

$$\int_a^b g(t, x) dt < l_k \quad \text{for } x > M, \quad (6)$$

where M is some positive constant, and

$$g(t, x) = \sup_{x \exp(-l_k) \leq y \leq x} \left\{ \sum_{i=1}^n \left[\frac{g_i(t, y^k)}{y} \right]^k \right\}^{1/k}.$$

Then problem (1)–(2) has at least one solution.

Condition (6) is satisfied, for example, if

$$\left\{ \sum_{i=1}^n [g_i(t, |x|^k)]^k \right\}^{1/k} \leq g(t)|x| + h(t, |x|),$$

$$g(t) \geq 0 \quad \text{for } a \leq t \leq b, \quad \int_a^b g(t) dt < l_k,$$

and the function $h(t, x) \in K(a, b)$ is nonnegative, nondecreasing in x , and

$$\lim_{x \rightarrow +\infty} \frac{1}{x} \int_a^b h(t, x) dt = 0.$$

If, in addition, one assumes that $f_i(t, x_1, \dots, x_n) \in K(a, b)$ ($i = 1, 2, \dots, n$), $k = 2$ and, instead of (5), the inequalities

$$|f_i(t, x_1, \dots, x_n)| \leq g_i \left(t, \sum_{j=1}^n |x_j|^2 \right) \quad (i = 1, 2, \dots, n),$$

hold, then we obtain the theorem of Lasota–Olech ⁽²⁾.

Theorem 2. Suppose that for $(t, x_1, \dots, x_n), (t, y_1, \dots, y_n) \in R_{ab}$ the inequalities

$$\begin{aligned} [f_i(t, x_1, \dots, x_n) - f_i(t, y_1, \dots, y_n)] \operatorname{sign}[(t - t_i)(x_i - y_i)] &\leq \\ &\leq g_i(t, |x_1 - y_1|, \dots, |x_n - y_n|) \quad (i = 1, 2, \dots, n), \end{aligned}$$

hold, where the functions $g_i(t, x_1, \dots, x_n) \in K(a, b)$ ($i = 1, 2, \dots, n$) are nonnegative and nondecreasing in each x_j , $1 \leq j \leq n$. Moreover, there do not exist absolutely continuous functions $\rho_i(t)$ ($i = 1, 2, \dots, n$) on the interval $[a, b]$, among which at least one is not identically equal to zero and which would satisfy conditions (4). Then problem (1)–(2) has at most one solution.

Corollary 1. Suppose that for $(t, x_1, \dots, x_n), (t, y_1, \dots, y_n) \in R_{ab}$ the inequalities

$$\begin{aligned} [f_i(t, x_1, \dots, x_n) - f_i(t, y_1, \dots, y_n)] \operatorname{sign}[(t - t_i)(x_i - y_i)] &\leq \\ &\leq g_i(t)|x_i - y_i| + h_i(t) \sum_{j=1}^n |x_j - y_j| \quad (i = 1, 2, \dots, n), \end{aligned}$$

hold, where the functions $g_i(t) \in L(a, b)$ and $h_i(t) \in L(a, b)$ ($i = 1, 2, \dots, n$) are nonnegative and

$$\max_{a \leq t \leq b} \sum_{i=1}^n \left| \int_{t_i}^t g_i(\tau) \exp \left(\int_{\tau}^t h_i(s) ds \right) d\tau \right| < 1.$$

Then problem (1)–(2) has at most one solution.

Corollary 2. Suppose that for $(t, x_1, \dots, x_n), (t, y_1, \dots, y_n) \in R_{ab}$ the inequalities

$$\begin{aligned} [f_i(t, x_1, \dots, x_n) - f_i(t, y_1, \dots, y_n)] \operatorname{sign}[(t - t_i)(x_i - y_i)] &\leq \\ &\leq g_i(t) \left\{ \sum_{j=1}^n |x_j - y_j|^k \right\}^{1/k} \quad (i = 1, 2, \dots, n), \end{aligned}$$

hold, where $1 < k < +\infty$, and the functions $g_i(t) \in L(a, b)$ ($i = 1, 2, \dots, n$) are nonnegative and

$$\int_a^b \left\{ \sum_{i=1}^n [g_i(t)]^k \right\}^{1/k} dt < l_k.$$

Then problem (1)–(2) has at most one solution.

According to Corollaries 2 and 3 of Theorem 1, it is easy to conclude that if $f(t, 0, \dots, 0) \in L(a, b)$, $f_i(t, x_1, \dots, x_n) \in K(a, b; t_i)$ ($i = 1, 2, \dots, n$) and either the conditions of Corollary 1 or the conditions of Corollary 2 of Theorem 2 are satisfied, then problem (1)–(2) has one and only one solution.

Theorem 3. Suppose that

$$f_i(t, |t - t_1|x_1, \dots, |t - t_n|x_n) \in K(a, b; t_i),$$

$$\text{vrai max}_{a \leq t \leq b} |f_i(t, 0, \dots, 0)| < +\infty \quad (i = 1, 2, \dots, n)$$

and, for $(t, x_1, \dots, x_n), (t, y_1, \dots, y_n) \in R_{ab}$, the inequalities

$$\begin{aligned} & [f_i(t, x_1, \dots, x_n) - f_i(t, y_1, \dots, y_n)] \text{sign}[(t - t_i)(x_i - y_i)] \leq \\ & \leq \alpha_i \left| \frac{x_i - y_i}{t - t_i} \right| + \beta_i \sum_{j=1}^n \left| \frac{x_j - y_j}{t - t_j} \right| \quad (i = 1, 2, \dots, n), \end{aligned}$$

where $0 \leq \alpha_i, \beta_i < 1$ ($i = 1, 2, \dots, n$) and

$$u \sum_{i=1}^n \frac{\beta_i}{1 - \alpha_i} < 1.$$

Then system (1) has one and only one solution $x_1(t), \dots, x_n(t)$, satisfying the boundary conditions

$$\overline{\lim}_{t \rightarrow t_i} \left| \frac{x_i(t)}{t - t_i} \right| < +\infty \quad (i = 1, 2, \dots, n).$$

Tbilisi State University

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REFERENCES

1. R. Conti, Math. Nachr., **23**, 161 (1961).
2. A. Lasota, C. Olech, Ann. Polon. Math., **18**, No. 2, 131 (1966).
3. A. Lasota, Z. Opial, Ann. Polon. Math., **19**, No. 1, 13 (1967).
4. O. Nicoletti, Atti Acad. Sci. Torino, **33**, 746 (1897).
5. A. I. Perov, A. V. Kibenko, Izv. AN SSSR, ser. matem., **30**, No. 2, 249 (1966).
6. V. P. Skripnik, Matem. sborn., **55** (97), No. 4, 449 (1961).
7. M. Švec, Mat. Fyz. Sb. Slovensk. Acad. Vied. a Umeni, **2**, 3 (1952).

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