ON THE PROBLEM OF COEFFICIENTS

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Abstract

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MATHEMATICS

I. A. ALEKSANDROV, V. Ya. GUTLYANSKII

ON THE PROBLEM OF COEFFICIENTS

IN THE THEORY OF UNIVALENT FUNCTIONS

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1°. In the complex plane I, the set of values of the functional

$$I(f) = c_{n+1} \qquad (n = 1, 2, ...) \tag{1}$$

on the class S of holomorphic univalent functions in the disk $E = \{z: \ |z| < 1\}$,

$$f(z) = z + c_2 z^2 + ... + c_{n+1} z^{n+1} + ...$$

forms a closed disk K_n with center at the point $I = 0$. Finding the radius of this disk is one of the principal problems in the theory of univalent functions.

The boundary functions with respect to the functional (1) on the class S are solutions of the differential-functional equation

$$Q(f(z)) \left(\frac{zf'(z)}{f(z)} \right)^2 = N(z) + \overline{N}\left(\frac{1}{z} \right), \tag{2}$$

where

$$Q(w) = x \sum_{m=1}^{n} \{f^{m+1}\}_{n+1} w^{-m},$$

$$N(z) = x \left[\frac{n}{2} \{f\}_{n+1} + \sum_{m=1}^{n} m \{f\}_m z^{m-n-1} \right].$$

By $\{\varphi\}_k$ here and below we denote the coefficient of z^k in the expansion of the holomorphic function $\varphi(z)$ in a series in a neighborhood of the origin; $x = e^{i\alpha}$,

$-\pi < \alpha \leq \pi$, is the unit vector of the outward normal to K_n at the boundary point $I = \{f\}_{n+1}$.

The study of the geometric properties of the functions forming the set $\widetilde{S}(n, x)$ of solutions of equation (2) on the class S is the subject of the works ([1–5]).

It is known that, for fixed n ($n = 1, 2, ...$) and α, $-\pi < \alpha \leq \pi$, the set $\widetilde{S}(n, x)$ contains those $2n$ functions

$$f(z) = z/(1 - \varepsilon z)^2 = z + 2\varepsilon z^2 + ... + (n+1)\varepsilon^n z^{n+1} + ... , \tag{3}$$

for which $\varepsilon = \varepsilon_l$, $\varepsilon_l = e^{\alpha i/n} \varepsilon_l^{(2n)}$, where $\varepsilon_l^{(2n)} = e^{\frac{\pi i}{n} l}$ ($l = 0, ..., 2n-1$). Each of them contributes to K_n either the point $(n+1)x$, or the point $-(n+1)x$. Function (3) maps the disk E onto the plane with a ray removed, joining the point $w = -1/(4\bar{\varepsilon})$ with $w = \infty$ and, under continuation, passing through the point $w = 0$. $\widetilde{S}(1, x)$ consists only of functions of the form (3) with $\varepsilon = \pm x$.

2°. In this note we give new examples of solutions of equation (2) and find a lower estimate for the number of functions constituting $\widetilde{S}(n, x)$. Satisfying the necessary condition of extremality in the coefficient problem, the proposed solutions contribute interior points to K_n, which is easy to see by comparing their moduli with the moduli of the corresponding coefficient

functions (3). However, with respect to a sufficiently large part of their neighborhood in the class S, regarded as a metric space with a metric convergence in which is equivalent to uniform convergence inside the disk, all known solutions of equation (2), including the new ones, are extremal.

3°. We denote, as usual, by S_p ($p = 1, 2, ...$) the subclass of the class S consisting of functions

$$f(z) = z + c_{p+1}^{(p)} z^{p+1} + \cdots + c_{kp+1}^{(p)} z^{kp+1} + \cdots ,$$

possessing p-fold rotational symmetry. Obviously, $S_1 \equiv S$.

Taking p ($p = 2, 3, ...$) to be one of the divisors of the number n ($n = 2, 3, ...$) and putting $n = kp$, consider the problem of determining the range of values of the functional

$$I(f) = c_{kp+1}^{(p)}$$

on the class S_p. Every function contributing a boundary point to this domain satisfies the equation

$$Q_p(f(z))\,(zf'(z)/f(z))^2 = N_p(z) + \overline{N}_p(1/z), \tag{4}$$

where

$$Q_p(w) = x\sum_{s=1}^{k}\{f^{sp+1}\}_{kp+1}w^{-sp}, \qquad x = e^{i\alpha}, \quad -\pi < \alpha \le \pi,$$

$$N_p(z) = x\left[\frac{kp}{2}\{f\}_{kp+1} + \sum_{s=0}^{k-1}(sp+1)\{f\}_{sp+1}z^{p(s-k)}\right].$$

Theorem 1. Let p be a divisor of the number n ($n = 2, 3, \dots;\ p = 2, 3, \dots$), and let k be the quotient obtained by dividing n by p. Then all solutions of equation (4) belonging to S_p are contained in $\widetilde{S}(n, x)$.

The proof of the theorem can be carried out by directly verifying that every solution of equation (4) also satisfies equation (2).

Corollary 1. For any fixed n ($n = 1, 2, \dots$), the functions

$$f(z) = \frac{z}{(1 \pm xz^n)^{2/n}} = z \mp \frac{2}{n}xz^{n+1} + \dots \in \widetilde{S}(n, x).$$

Each of them maps the disk E onto the whole plane with slits removed, beginning at the points $w = \frac{1}{\sqrt[n]{4}}e^{-\alpha_i/n}e_l^{(n)}$ (or $w = \frac{1}{\sqrt[n]{4}}e^{\frac{\pi-\alpha}{n}i}e_l^{(n)}$),

$$e_l^{(n)} = e^{\frac{2\pi i}{n}} \qquad (l = 0, \dots, n-1),$$

and, upon continuation, passing through the point $w = 0$.

Corollary 2. Suppose that the number n admits a binary factorization $n = kp$ with factors k, p different from unity. Then

$$f(z) = \frac{z}{(1 - \varepsilon z^p)^{2/p}} = z + \dots + \binom{-2/p}{k}(-\varepsilon)^k z^{n+1} + \dots \in \widetilde{S}(n, x)$$

for $\varepsilon = e^{\alpha i/k}e_l^{(2k)}$ $(l = 0, \dots, 2k-1)$.

Corollary 3. In $\widetilde{S}(4, x)$, in addition to the functions indicated in Corollaries 1 and 2, there are the functions $f(z) = \eta\varphi(\eta z)$, $\eta = \pm e^{\frac{\alpha+k\pi}{2}i}$ $(k = 0, 1)$, where $w = \varphi(z)$ is implicitly defined by the equation

$$\frac{a}{2}\ln\frac{1+\sqrt{1-aw}}{1-\sqrt{1-aw}} - \frac{\sqrt{1-aw}}{w} = z - \frac{1}{z} - \frac{a}{2}\ln z, \qquad a = 4e^{-1}.$$

The function $\varphi(z)$ maps the disk E onto the domain obtained by removing from the plane the ray $w_0 \le \operatorname{Re} z \le +\infty$, $w_0 = 1/a$, and two analytic arcs symmetric with respect to the real axis, forming angles $2\pi/3$ with this ray at the point w_0.

The function $\varphi(z)$ is a solution of equation (4) for $k = p = 2$ and $x = \pm 1$.

Theorem 2. If the canonical factorization of the number n has the form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} \quad (p_1, p_2, \dots, p_k \text{ are prime numbers}), \tag{5}$$

then the set $\widetilde{S}(n, x)$ consists of no fewer than $2\beta(n)$ elements, where

$$\beta(n) = \prod_{r=1}^{k} \frac{p_r^{\alpha_r+1} - 1}{p_r - 1}.$$

In particular,

$$\beta(4) = 7, \quad \beta(5) = 6, \quad \beta(6) = 12, \quad \beta(7) = 8, \quad \beta(8) = 15.$$

Upper estimates for the number of elements in $\widetilde{S}(n, x)$ are unknown. The points obtained from $\widetilde{S}(n, x)$ by functions in K_n form isolated circular orbits $\widetilde{K}(n)$ with center at $I = 0$. Lower estimates for the number $\gamma(n)$ of such orbits are given by

Theorem 3. If n has the canonical factorization (5), then

$$\gamma(n) \geq 1 + (\alpha_1 + \alpha_2 + \cdots + \alpha_k) + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \cdots + \alpha_{k-1}\alpha_k) +$$

$$+ (\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \cdots + \alpha_{k-2}\alpha_{k-1}\alpha_k) + \cdots + \alpha_1\alpha_2 \cdots \alpha_k.$$

In particular,

$$\gamma(4) \geq 3, \quad \gamma(5) \geq 2, \quad \gamma(6) \geq 4, \quad \gamma(7) \geq 2, \quad \gamma(8) \geq 4.$$

Upper estimates for $\eta(n)$ are unknown.

4°. Analogous propositions hold for solutions of the functional-differential equations characterizing, in the method of interior variations, the boundary functions in the problem on the range of values of the functional

$$I(F) = b_n \tag{6}$$

on the class Σ of univalent functions in $|z| > 1$ of the form $F(z) = z + b_0 + b_1/z + \cdots + b_n/z^n + \cdots$, holomorphic in $1 < |z| < \infty$.

In view of the duality of the problem on the range of values of the functional (1), [(6)] on $S[\Sigma]$ to a certain functional depending polynomially on the coefficients of functions of the class $\Sigma[S]$, the non-uniqueness theorems indicated for the class $S[\Sigma]$ carry over, with natural changes, to the class $\Sigma[S]$.

In conclusion we note that the theorems on non-uniqueness of solutions of the equations constituting a necessary condition for extremality of a function in the method of interior variations also occur in more general problems connected with the determination of ranges of values of functionals analytically depending on the coefficients of a function $f(z)$ of the class S. We indicate only one of them, having first agreed to denote by $\widetilde{S}(n_1, \dots, n_l; x)$ the set of solutions of the differential-functional equation corresponding to the problem on the range of values of the functional

$$I(f) = J(c_{n_1+1}, \dots, c_{n_l+1}) \tag{7}$$

on the class S; $J = J(w_1, \dots, w_l)$ is an analytic function in the domain of variation of the coefficients $(c_{n_1+1}, \dots, c_{n_l+1})$, $f(z) \in S$, in the space C^l.

Theorem 4. Let p be a common divisor of the numbers $n_1, \dots, n_l$ $(l = 2, 3, \dots)$. Then all solutions of the equation characterizing boundary functions with respect to the functional (7) on the class S_p belong to $\widetilde{S}(n_1, \dots, n_l; x)$.

Tomsk State University
named after V. V. Kuibyshev

Donetsk Computing Center
of the Academy of Sciences of the Ukrainian SSR

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Note: Figure translations are in progress. See original paper for figures.