

ON THE SOLVABILITY OF CERTAIN CLASSES OF NONLINEAR EQUATIONS

MATHEMATICS

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.90930>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.948:513.88

MATHEMATICS

R. I. KACHUROVSKII

ON THE SOLVABILITY OF CERTAIN CLASSES OF NONLINEAR EQUATIONS

(Presented by Academician A. N. Tikhonov on 15 III 1969)

In the paper ⁽¹⁾, by means of a certain analogue of A. N. Tikhonov's fixed-point principle ⁽²⁾, the solvability of nonlinear equations with weakly continuous coercive operators was proved (see also ⁽³⁾). In § 1 of the present note these investigations are continued.

A theorem is obtained on the solvability of nonlinear equations with weakly continuous noncoercive operators (Theorem 2). The rejection of the coercivity condition is essential, for, as simple examples show, even linear operators in finite-dimensional spaces satisfying, for example, the conditions of Theorem 2 by no means always satisfy the coercivity condition. In § 2 a new class of operators is introduced. The solvability of equations with operators from this class is obtained.

1. Let X, Y be Banach spaces; $A : X \rightarrow Y$ a bounded linear operator defined on all of X ; A^* its adjoint. Let X^* be the space conjugate to X . It is known that if $\|A^*g\|_{X^*} \geq m\|g\|_{Y^*}$, $\forall g \in Y^*$, where the constant number $m > 0$, then the equation $Ax = y$ is solvable for every $y \in Y$. Theorems 1 and 2, in a certain sense, generalize this fact to some classes of nonlinear operators.

Theorem 1 (cf. ⁽⁹⁾). *Let X be a separable Banach space, $Y = X^*$ the space conjugate to X . Suppose the norm in the space Y is Fréchet differentiable* at every point $y \in Y$, $y \neq 0$. Let $F : X \rightarrow Y$ be an operator defined on the whole space X , having a weakly closed range** and having at each point $x \in X$ a Fréchet derivative $F'(x)$ satisfying the condition*

$$\|[F'(x)]^*v\|_{X^*} \geq k(x)\|v\|_{Y^*}, \quad \forall v \in Y^*, \forall x \in X,$$

where the functional $k(x) > 0$ for all $x \in X$. Then the range of the operator F is the whole space Y .

In what follows weak convergence is denoted by the sign \rightharpoonup , and strong convergence by the sign \rightarrow . Let us recall the known definitions. An operator

$F : X \rightarrow Y$ is called semicontinuous (weakly continuous) in X if from $x_n \rightarrow x_0$ ($x_n \rightharpoonup x_0$) in X it follows that $F(x_n) \rightarrow F(x_0)$ in Y as $n \rightarrow \infty$.

Theorem 2. Let X be a reflexive Banach space, Y a Banach space. Let $F : X \rightarrow Y$ be a weakly continuous operator defined on all of X , satisfying the condition $\lim_{\|x\| \rightarrow \infty} \|F(x)\| = \infty$. Suppose that for every $x \in X$ there exists a Fréchet derivative $F'(x)$, satisfying the condition

$$\|[F'(x)]^*v\|_{X^*} \geq k(x)\|v\|_{Y^*}, \quad \forall v \in Y^*, \forall x \in X,$$

where the functional $k(x) > 0$ for all $x \in X$.

Then the range of the operator F is the whole space Y .

* A bounded linear operator $T'(x_0) : X \rightarrow Y$ is called the Fréchet derivative of a linear or nonlinear operator $T : X \rightarrow Y$ at the point $x_0 \in X$ if for all $h \in X$ the equality

$$\lim_{\|h\| \rightarrow 0} \frac{\|T(x_0 + h) - T(x_0) - T'(x_0)h\|}{\|h\|} = 0 \quad (\text{see } (4)).$$

holds.

** That is, from the weak convergence of any sequence $F(x_n)$ to $y \in Y$ follows the existence of an element $x_0 \in X$ such that $y = F(x_0)$.

2. Let E be a Banach space; E^* its conjugate; $F : E \rightarrow E^*$ an operator defined on all of E . The notation (z, x) denotes the value of the linear functional $z \in E^*$ on the element $x \in E$.

Definition. We shall say that the operator F belongs to the class K if, for any $x, h \in E$, from the inequality $(F(x), h) \geq 0$ there follows the inequality $(F(x+h), h) \geq 0$.

Obviously, every monotone operator (see (6)) belongs to the class K . The following lemmas give other sufficient conditions for an operator to belong to the class K .

Lemma 1. Let H be a Hilbert space; let $\mu(x)$ be a functional defined on all of H ; $\mu(x) > 0, \forall x \in H$.

For $E = E^* = H$, every operator $F : H \rightarrow H$ of the form $F(x) = \mu(x)x$ belongs to the class K .

Lemma 2. Let R^1 be the real line, $E = E^* = R^1$. Suppose the function $F : R^1 \rightarrow R^1$ satisfies at least one of the following conditions:

a) $F(x) > 0, \forall x \in R^1$; b) $F(x) < 0, \forall x \in R^1$; c) $F(x)$ is a continuous function, nondecreasing on the sets $I_1 = (-\infty, a)$ and $I_3 = (b, +\infty)$, and on the segment $I_2 = [a, b], a < b$, the number $\inf_{x \in I_2} F(x) > 0$.

Then F is a function of the class K .

From these lemmas it is clear that the class K is considerably broader than the class of monotone operators, which has recently been studied in the works of many authors, in particular in (5–8).

It is interesting that in the case when the operator F is representable in the form $F(x) = Ax - \varphi$, where $\varphi \in E^*$ is a fixed element, and A is a linear operator defined on all of E with values in E^* , we do not obtain an extension of the class of monotone operators. This follows from Lemma 3.

Lemma 3. If the operator $F(x) = Ax - \varphi$ belongs to the class K for some fixed φ , then A is a positive operator, i.e. $(Ah, h) \geq 0, \forall h \in E$.

We now turn to the question of solvability of equations with operators from the class K .

Theorem 3. Let E be a reflexive Banach space; let $F : E \rightarrow E^*$ be a semi-continuous operator of the class K , defined on all of E , satisfying the condition

$$(F(x), x) \geq \gamma(\|x\|)\|x\|, \quad \forall x \in E,$$

where γ is a real function of $t \geq 0, \lim_{t \rightarrow +\infty} \gamma(t) = +\infty$.

Then the equation $F(x) = 0$ is solvable in E , and the set of all solutions of this equation is a weakly closed convex set.

Theorem 4. Let E be a reflexive Banach space; let $F : E \rightarrow E^*$ be a semi-continuous operator of the class K , defined on all of E , satisfying the condition: F is the gradient of a real-valued functional φ defined on the whole space E ,

$$\lim_{\|x\| \rightarrow \infty} \varphi(x)/\|x\| = +\infty.$$

Then the range of the operator F is the entire space E^* .

A generalization of the last theorem is the following result.

Theorem 5. Let E be a reflexive Banach space, $F : E \rightarrow E^*$ an operator representable in the form $F = T_1 + T_2 + T_3$. Suppose the following conditions are fulfilled: a) the operators $T_i, i = 1, 2, 3$, are defined on all of E and are semi-continuous, T_1 is an operator of the class K , T_2 is monotone, T_3 is compact; b) the operators $T_i, i = 1, 2, 3$, are gradients of single-valued real-valued functionals φ_i defined on all of E , and the functional

$$\varphi(x) = \varphi_1(x) + \varphi_2(x) + \varphi_3(x)$$

satisfies the condition

$$\lim_{\|x\| \rightarrow \infty} \varphi(x)/\|x\| = +\infty.$$

Then the range of the operator F is the entire space E^* .

We formulate a lemma that may prove useful for various extremum problems.

Lemma 4. Let D be a bounded closed convex set in a reflexive Banach space E ; $\varphi(x)$ is a functional defined on D , whose gradient is a hemicontinuous operator of class K (more precisely, from the conditions $x \in D$, $(x+h) \in D$, $(F(x), h) \geq 0$ it follows that $(F(x+h), h) \geq 0$, where $F(x) = \text{grad } \varphi(x)$).

Then there exists at least one point $x_0 \in D$ such that

$$\varphi(x_0) = \inf_{x \in D} \varphi(x).$$

Let us also note that the assertion of Theorem 3 remains valid if the requirement that the operator F belong to the class K is replaced by the following: for any $x, h \in E$, from the equality $(F(x), h) = 0$ it follows that $(F(x+h), h) \geq 0$.

All-Union Correspondence
Electrotechnical Institute of Communications
Moscow

Received
6 III 1969

CITED LITERATURE

- (¹) R. I. Kachurovskii, DAN, **183**, No. 3 (1968).
- (²) A. N. Tikhonov, Math. Ann., **111**, 767 (1935).
- (³) R. I. Kachurovskii, Inform. byull. No. 7, International Congress of Mathematicians, Moscow, 1966.
- (⁴) M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.
- (⁵) J. L. Lions, Bull. Soc. Math. France, **93**, 155 (1965).
- (⁶) F. E. Browder, Proc. Nat. Acad. Sci. U.S.A., **56**, No. 2, 419 (1966).
- (⁷) W. V. Petryshyn, Trans. Am. Math. Soc., **126**, No. 1, 43 (1967).
- (⁸) H. Brezis, C. R., Ser. A, **264**, No. 15, 683 (1967).
- (⁹) S. I. Pokhozhaev, DAN, **184**, No. 1, 40 (1969).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.