MORSE THEORY OF NONMULTIPLE CLOSED CURVES

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MORSE THEORY OF NONMULTIPLE CLOSED CURVES

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As is known, the application of Morse theory to the question of the number of geometrically distinct closed geodesics encounters a difficulty connected with the need to distinguish multiple geodesics from simple (nonmultiple) ones. The natural idea—to consider only nonmultiple curves—takes us outside the framework of classical Morse theory, since the space of nonmultiple closed curves is not complete. This latter circumstance gives rise to pessimism, since, generally speaking, Morse theory on incomplete manifolds is meaningless in view of the fact that topological changes here may arise only at the expense of incompleteness, i.e., not depend on the critical points of the function under consideration. Fortunately, the space of nonmultiple curves has the form $M \setminus K$, where M is a complete space and K is its closed subset. Below we show that in such a situation, if the set K is sufficiently "well behaved," a meaningful Morse theory is possible.

In this note we first sketch an outline of Morse theory for manifolds of the form $M \setminus K$, and then give the results of applying this theory to the space of nonmultiple curves and to the problem of closed geodesics. As another application of our theory, we compute the homotopy type of the space of embeddings of one surface into another (in particular, the type of the space of diffeomorphisms of an arbitrary surface).

1. General considerations. Let M be a complete Riemannian manifold (not necessarily finite-dimensional), f a smooth function on it, and $K \subset M$ some closed subset. A homotopy $h : L \times I \to M$ will be called monotone with respect to f if, for any point $y \in L$,
$f(h(y,t_1)) > f(h(y,t_2))$ whenever $t_1 < t_2$.

Definition. A point $x \in K \cap \overline{(M \setminus K)}$ is called **responsible with respect to the function** f if there exist a neighborhood $U \ni x$ and a number $\varepsilon = \varepsilon(U)$ such that the set $U \setminus K$ can be monotonically deformed into itself in
$f^{-1}([-\infty, f(x) - \varepsilon]) \cap (U \setminus K)$.

Thus the points of the closed set $K \cap \overline{(M \setminus K)}$ are divided into two classes: responsible with respect to f and nonresponsible. It is obvious that the set

$V = V(f, K)$ of responsible points is open in $K \cap \overline{(M \setminus K)}$.

A neighborhood U possessing the properties formulated in the definition of a responsible point will be called **admissible**.

Definition. Let $S \subset V$ be some set. We shall call S **uniformly responsible with respect to** f if there exist a collection of admissible neighborhoods $\{U_i\}$ and a number $\varepsilon > 0$ such that $S \subset \bigcup_i U_i$ and $\varepsilon(U_i) > \varepsilon$ for all i.

Example. $M = R^2$, $K_1 = \{x = 0\}$, $K_2 = \{xy = 1\}$, $K_3 = \{x^2 + y^2 = 1\}$, $f(x, y) = y$. Then all points of K_1 are responsible and uniformly responsible with respect to f. The points of the set K_2 are responsible, but not uniformly responsible with respect to f. Finally, the points $(0, \pm 1) \in K_3$ are nonresponsible with respect to f, whereas the remaining points are responsible. $V(f, K_3)$ is not uniformly ···

noncritical, while $V(f, K_3) \setminus U$, where U is an arbitrary neighborhood of the point $(0, -1)$, is uniformly noncritical.

Let

$$V(t_1, t_2) = V(f, K, t_1, t_2) = V(f, K) \cap f^{-1}([t_1, t_2]).$$

The following lemma underlies the proposed method:

Fundamental Lemma. Let

$$(M \setminus K)_t = (M \setminus K) \cap f^{-1}([-\infty, t)).$$

Then the inclusion

$$(M \setminus K)_{t_1} \subset (M \setminus K)_{t_2}$$

is a weak homotopy equivalence if the set $V(t_1, t_2)$ is uniformly noncritical.

Remark 1. In fact, in practically interesting cases, under the hypotheses of Lemma 1 one can show that $(M \setminus K)_{t_1}$ is a deformation retract of $(M \setminus K)_{t_2}$.

Remark 2. Every compact set in $V(f, K)$ is uniformly noncritical.

Definition. We shall call the set K *admissible with respect to* f if the sets

$$V(f, K) \cap U \cap f^{-1}([-\infty, t])$$

are uniformly noncritical for every t whenever U is an open set in M containing all noncritical points of the set K.

The fundamental lemma shows that, in the case where the critical values of the function f, say $\{t_i\}_{i \in \omega}$, and the images of noncritical points

$$\{s_i\}_{i \in \omega} = f(\{(\overline{M \setminus K}) \cap K\} \setminus V)$$

together form a discrete set in R, and the set K is admissible with respect to f, it is possible to study the topology of the manifold $M \setminus K$ in the following way.

Consider the family of manifolds $(M \setminus K)_t,\ t \in R$. Then the homotopy type of the manifold $(M \setminus K)_t$ changes when passing either through a critical value t_i, or through a noncritical value s_j. Therefore, if we know what is attached to the manifold $(M \setminus K)_t$ when passing through critical and noncritical values, the homotopy topology of the manifold $M \setminus K$ is known to us.

In what follows, we shall call the method described above the *method of noncritical points*. Below we apply it to the space of nonmultiple curves.

The space of nonmultiple curves. Let P be a complete Riemannian manifold, $M' = M'(P)$ the space of all smooth mappings $S^1 \to P$, f' the length function on M', and $K' \subset M'$ the set of all multiple curves in M'. (A curve $\alpha : S^1 \to P$ is called multiple if $\alpha = \alpha' \circ \beta_m$, where $\beta_m : S^1 \to S^1$ is the canonical mapping of degree $m \neq \pm 1$.)

Proposition 1. a) K' is admissible in M' with respect to f', if $\pi_1(P) = 1$; b) a point $x \in K'$ is noncritical if and only if $\operatorname{grad}_x f' = 0$; c) under the hypotheses of the fundamental lemma there is a homotopy equivalence

$$(M' \setminus K')_{t_1} \subset (M' \setminus K')_{t_2}.$$

Thus the critical points of the function f' on $M' \setminus K'$ correspond to simple closed geodesics, while the noncritical points correspond to multiple closed geodesics.

Proposition 2. To the manifold $(M' \setminus K')_t$, when passing through an isolated noncritical point corresponding to a multiple geodesic g of index λ, from the homotopy point of view there is attached the space

$$D^\lambda \times O(2)$$

according to some mapping

$$\partial D^\lambda \times O(2) \to (M' \setminus K')_t.$$

Let us note that the natural embedding $(M' \setminus K') \to M'$ is (because K' has infinite codimension in M') a weak homotopy equivalence. This fact, together with Proposition 2, shows that the study of the space $M' \setminus K'$ for the theory of simple closed geodesics is apparently unpromising. In this connection we pass to the consideration of the spaces

$$M = M'/O(2), \qquad K = K'/O(2)$$

and $M \setminus K$.

The space M, however, is a manifold with singularities, and its set of singularities coincides with K, so that $M \setminus K$ is an open manifold without singularities. The method of noncritical points described above can be applied in this case (despite the fact that M has singularities). Denote by f the function induced on M by the function f'. Then the following holds:

Proposition 3. a) K is admissible in M with respect to f, if $\pi_1(P) = 1$; b) a point $x \in K$ is nondegenerate only when it is a multiple closed geodesic; c) under the hypotheses of the main lemma there is a homotopy equivalence

$$(M \setminus K)_{t_1} \subset (M \setminus K)_{t_2}.$$

Let now g be a simple closed geodesic, and g_m its m-th multiple. Denote by $\lambda(h)$ the index of the closed geodesic h (in M'). Fix g and put $\lambda = \lambda_1 = \lambda(g)$, $\lambda_m = \lambda(g_m)$. Obviously, $\lambda_1 \leq \lambda_m$, $m = 1, 2, \ldots$. Let $L_m = K(Z_m, 1)$, $\varphi_m : Z_m \to SO(\lambda_m)$ be some representation of the group Z_m; $\xi_m = \xi(\varphi_m)$ the vector λ_m-dimensional bundle over L_m determined by φ_m; $T_m = T(\xi_m)$ the bundle of spheres D^{λ_m} corresponding to ξ_m. The following theorem forms the basis of our approach to the theory of closed geodesics.

Theorem 1. Let the geodesic g_m, $m > 1$, be nondegenerate. Then: a) the multiplicity of the trivial representation entering φ_m is equal to λ, and

$$\ker \varphi_m = 0;$$

b) from the homotopy point of view, passage through the isolated nondegenerate point g_m corresponds to attaching to $(M \setminus K)_t$ the space T_m according to some map

$$\partial T_m \to (M \setminus K)_t.$$

Closed geodesics. Theorem 1 shows that, in the space $M \setminus K$, passage through a multiple geodesic differs essentially from passage through a simple one. The fact that, when passing through a multiple geodesic, an infinite number of cells specifically related to one another is attached to the region of smaller values from the homotopy point of view makes it possible in a number of cases to establish the infinitude of the number of simple closed geodesics.

Below we describe a general scheme of reasoning which, in some concrete cases, leads to establishing the infinitude of the number of simple closed geodesics.

A. Let $M'_0 \setminus K'$ be the space of nonmultiple curves of "small" length, which, as is not difficult to see, is homotopy equivalent to the manifold P, if $\pi_1(P) = 1$. Therefore one can, generally speaking, compute the cohomology of the space

$$M_0 \setminus K = M'_0 \setminus K'/O(2),$$

by considering the principal fibration

$$P \sim M'_0 \xrightarrow{O(2)} M_0 \sim M_0 \setminus K.$$

B. Since M' is homotopy equivalent to $M' \setminus K'$, one can, generally speaking, compute the cohomology of the space $M' \setminus K'$, by considering the fibration

$$\theta : M' \to P, \qquad \text{where } \theta(f) = f(1)$$

(here $f : S^1 \to P$, $f \in M'$, and

$$S^1 = \{z \in C \mid |z| = 1\}.$$

Further, generally speaking, one can compute the cohomology of the space $M \setminus K$, by considering the principal fibration

$$M' \setminus K' \to M \setminus K.$$

C. Using the results of A and B, one can, generally speaking, compute the cohomology of the pair

$$(M \setminus K,\, M_0 \setminus K).$$

D. Applying Proposition 3 and Theorem 1 together with the results of [1], which establish the interrelation between the numbers λ_m, $m = 1, 2, \ldots$, one can sometimes become convinced that, because of multiple geodesics, the pair $(M \setminus K,\, M_0 \setminus K)$ has far more cohomology than is "necessary" by C. Therefore the "extra cohomologies," which arise very irregularly, can be eliminated, by Theorem 1 and the properties of the sequence $\{\lambda_m\}$, only by admitting the existence of an infinite number of simple geodesics.

In conclusion we note that what has been said above, together with some other considerations, leads evidently to the following hypothesis.

Hypothesis. On every compact Riemannian manifold the closed geodesics are everywhere dense, i.e. the set of tangent vectors tangent to closed geodesics is everywhere dense in the tangent space to P.

Spaces of embeddings of a circle. Let P be a smooth manifold. $\mathrm{Im}(S^1, P)$ is the space of all immersions $S^1 \to P$, and

$$\mathrm{Pl}(S^1, P) \subset \mathrm{Im}(S^1, P)$$

is its subspace consisting of embeddings.

Consider on P some Riemannian metric g, and consider in $\mathrm{Im}(S^1, P)$ the subspace $\mathrm{Im}_g(S^1, P)$, consisting of curves parametrized by arc length. As is known, Im_g is a deformation retract of the space Im. The same is true also for

$$\mathrm{Pl}_g(S^1, P) = \mathrm{Im}_g(S^1, P) \cap \mathrm{Pl}(S^1, P).$$

Let $K_g = \mathrm{Im}_g - \mathrm{Pl}_g$. Obviously, K_g is closed in Im_g.

Proposition 4. *The only non-hanging points of the set K_g in Im_g relative to the length function are self-intersecting closed geodesics.*

Proposition 4, in the case when P has a "good"metric, makes it possible to obtain information about the homologies of the space Pl_g. If $\dim P = 2$, then, as is known, one can always introduce on P a complete metric of constant curvature. In this case Proposition 4 easily leads to the following theorem.

Theorem 2. *The space* $\mathrm{Pl}(S^1, P)$*,* $\dim P = 2$*, is homotopy equivalent to its subspace consisting of curves of constant geodesic curvature.*

From this theorem, in turn, it follows easily that

Theorem 3. *Let* P *be a surface (not necessarily compact) endowed with a complete metric of constant curvature. Then the space of its diffeomorphisms* $\mathrm{Diff}\, P$ *is homotopy equivalent to the space of its isometries* $\mathrm{Iso}(P)$*.*

The results formulated in these theorems are mostly known. In conclusion we note the following:

Proposition 5. *The set* $M' \setminus \mathrm{Im}(S^1, P)$ *is openly situated in* M'*, but it is not admissible.*

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References

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Note: Figure translations are in progress. See original paper for figures.