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Abstract

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IMPLICIT FUNCTIONS AND THE AVERAGING PRINCIPLE OF N. N. BOGOLYUBOV–N. M. KRYLOV

(Presented by Academician N. N. Bogolyubov, 27 V 1968)

In recent years interest in the theory of implicit functions has increased. As is well known, the conditions of the classical theorems on implicit functions contain substantial assumptions on the smoothness of the corresponding operators. These assumptions exclude from consideration the most interesting and important applications. For example, the classical theorem of N. N. Bogolyubov on averaging on an infinite interval clearly has the character of a theorem on an implicit function, but it does not follow from the standard theorems on implicit functions.

Below we propose one generalization of the theorem on implicit functions, from which there immediately follows the averaging principle on an infinite interval under more general conditions than those indicated by N. N. Bogolyubov and his students.

1. Let Λ, E, F be Banach spaces. Suppose that for $\|\lambda\| \le a$, $\|x\| \le b$ there is defined an operator $f(\lambda, x)$ with values in F, satisfying the condition $f(0,0) = 0$. We shall assume that for all λ from the ball $\|\lambda\| \le a$ there exists the Fréchet derivative $f'_x(\lambda, 0)$ of the operator $f(\lambda, x)$ and, moreover, that the condition

$$\|f(\lambda, x_1) - f(\lambda, x_2) - f'_x(\lambda, 0)(x_1 - x_2)\| \le q(\rho, r)\|x_1 - x_2\| \tag{1}$$

$$(\|\lambda\| \le \rho;\ \|x_1\|,\ \|x_2\| \le r),$$

is fulfilled, where $q(\rho, r)$ $(q(0,0) = 0)$ is a function continuous at zero.

Theorem 1. *Suppose the following conditions are fulfilled:*

a) $\lim\limits_{\lambda \to 0} \|f(\lambda, 0)\| = 0$;

b) *for all* λ *from the ball* $\|\lambda\| \le a$ *there exists an operator* $\Gamma(\lambda) = [f'_x(\lambda, 0)]^{-1}$, *and*

$$\|\Gamma(\lambda)\| \leq c \quad (\|\lambda\| \leq a). \tag{2}$$

Then there exist positive numbers a_0, b_0 *such that, for* $\|\lambda\| \leq a_0$*, the equation* $f(\lambda, x) = 0$ *has in the ball* $\|x\| \leq b_0$ *a unique solution* $x = x(\lambda)$*, and* $\|x(\lambda)\| \to 0$ *as* $\|\lambda\| \to 0$*.*

For the proof it is necessary to consider the operator

$$\Pi(\lambda, x) = x - \Gamma(\lambda) f(\lambda, x). \tag{3}$$

In view of (1) and (2), this operator satisfies the Lipschitz condition

$$\|\Pi(\lambda, x_1) - \Pi(\lambda, x_2)\| \leq cq(\rho, r)\|x_1 - x_2\| \quad (\|\lambda\| \leq \rho;\ \|x_1\|,\ \|x_2\| \leq r).$$

Hence, in particular, the inequality follows

$$\|\Pi(\lambda, x)\| \leq cq(\rho, r)\|x\| + c\|f(\lambda, 0)\| \quad (\|\lambda\| \leq \rho,\ \|x\| \leq r).$$

Choose the numbers a_0 and b_0 so that for $\|\lambda\| \leq a_0$ the inequalities $cq(a_0, b_0) = q < 1$, $c\|f(\lambda, 0)\| \leq (1 - q)b_0$ are fulfilled. Then the operator (3), for $\|\lambda\| \leq a_0$, satisfies on the ball $\|x\| \leq b_0$ the conditions of the contraction mapping principle.

2. In what follows, R denotes a finite-dimensional space; the norm of an element $x \in R$ and the norm of a linear operator C acting in R are denoted respectively by $|x|$ and $|C|$. Let $\mathcal{C}$ be the space of functions $x(t)$, continuous and bounded on the entire axis, with values in R, with the usual norm.

Let B be a matrix having no zero or purely imaginary eigenvalues, and let $G_0(t, s)$ be the Green' s function for the problem of bounded solutions of the equation

$$dx/dt = Bx + f(t). \tag{4}$$

Lemma. Let $f(t)$ be a bounded continuous function. Then the equality

$$\lim_{T \to \infty} \left\| \frac{1}{T} \int_t^{t+T} f(s)\, ds \right\|_{\mathcal{C}} = 0 \tag{5}$$

is equivalent to the equality

$$\lim_{\varepsilon \to \infty} \left\| \int_{-\infty}^{\infty} G_0(t, s) f\left(\frac{s}{\varepsilon}\right)\, ds \right\|_{\mathcal{C}} = 0. \tag{6}$$

Suppose that we are given a bounded and continuous matrix-function $A(t)$ $(-\infty < t < \infty)$. We shall consider the problem of bounded solutions for the differential equation

$$dx/dt = A(t/\varepsilon)x + f(t). \tag{7}$$

Theorem 2. Let

$$\lim_{T\to\infty} \left\| \frac{1}{T} \int_t^{t+T} [A(s) - B]\, ds \right\|_{\mathcal{C}} = 0. \tag{8}$$

Then there exists an $\varepsilon_0 > 0$ such that, for $0 < |\varepsilon| \leq \varepsilon_0$, the problem of bounded solutions of equation (7) has a Green' s function $G_\varepsilon(t, s)$, and as $\varepsilon \to 0$ the integral operators G_ε with kernels $G_\varepsilon(t, s)$ converge in norm to the integral operator G_0 with kernel $G_0(t, s)$.

For the proof, consider the matrix

$$H(t, \varepsilon) = \int_{-\infty}^{\infty} G(t, \tau) \left[A\left(\frac{\tau}{\varepsilon}\right) - B \right] d\tau.$$

It follows from the lemma that, as $\varepsilon \to 0$, the norm $|H(t, \varepsilon)|$ tends uniformly to zero. After the substitution $x = y + H(t, \varepsilon)y$, equation (7) becomes the system

$$dy/dt = By + D(t, \varepsilon)y + [I + H(t, \varepsilon)]^{-1} f(t), \tag{9}$$

where $D(t, \varepsilon)$ tends uniformly to zero as $\varepsilon \to 0$. Therefore there exists an $\varepsilon_0 > 0$ such that, for $0 < |\varepsilon| \leq \varepsilon_0$, equation (9) has a unique bounded solution

$$y_\varepsilon(t) = \overline{G}_\varepsilon f_\varepsilon(t),$$

where

$$f_\varepsilon(t) = [I + H(t, \varepsilon)]^{-1} f(t)$$

and

$$\overline{G}_\varepsilon f(t) = \int_{-\infty}^{\infty} \overline{G}_\varepsilon(t, s) f(s)\, ds,$$

with $\|\overline{G}_\varepsilon - G_0\| \to 0$ as $\varepsilon \to 0$. Thus equation (7) also has a unique bounded solution

$$x_\varepsilon(t) = G_\varepsilon f(t),$$

where

$$G_\varepsilon f(t) = \int_{-\infty}^{\infty} G_\varepsilon(t,s) f(s)\, ds;$$

here

$$G_\varepsilon(t,s) = [I + H(t,\varepsilon)]\overline{G}_\varepsilon(t,s)[I + H(s,\varepsilon)]^{-1}.$$

It can be shown that, for some $\delta > 0$, the Green functions $G_\varepsilon(t,s)$ of equation (7) converge to the Green function $G_0(t,s)$ uniformly with weight $e^{\delta|t-s|}$.

Theorem 2 admits a converse.

Theorem 3. *Suppose that, for sufficiently small ε, equation (7) has a unique bounded solution $x_\varepsilon(t) = G_\varepsilon f(t)$ for every bounded continuous function $f(t)$. Suppose that, as $\varepsilon \to 0$, the operators G_ε converge, in the norm of operators acting in $\mathcal{C}$, to the operator G_0.*

Then equality (8) holds.

3. **Theorem 4.** *Let $X(t,x)$ $(-\infty < t < \infty,\ |x| \le a)$ be an operator with values in R, continuous in t and uniformly continuous with respect to t in x; let there exist the Fréchet derivative $A(t) = X_x(t,0)$, and let, for $-\infty < t < \infty$, $|x_1|, |x_2| \le r \le a$, the inequality*

$$\left|X(t,x_1) - X(t,x_2) - A(t)(x_1 - x_2)\right| \le \omega(r)|x_1 - x_2|. \tag{10}$$

hold, where $\omega(r) \to 0$ as $r \to 0$. Suppose that the matrix B has neither zero nor purely imaginary eigenvalues. Finally, suppose that, uniformly with respect to t, the equalities

$$\lim_{T\to\infty} \left|\frac{1}{T}\int_t^{t+T} X(s,0)\, ds\right| = 0, \tag{11}$$

$$\lim_{T\to\infty} \left|\frac{1}{T}\int_t^{t+T} [A(s) - B]\, ds\right| = 0. \tag{12}$$

hold. Then there exist $a_0, \varepsilon_0 > 0$ such that, for $0 < |\varepsilon| \le \varepsilon_0$, the equation

$$dx/dt = \varepsilon X(t,x) \tag{13}$$

has a unique solution $x_\varepsilon(t)$, lying in the ball $|x| \le a_0$ for all t, and

$$\lim_{\varepsilon\to 0} \|x_\varepsilon(t)\|_{\mathcal{C}} = 0. \tag{14}$$

Proof. Equation (13) (for $\varepsilon \ne 0$) is equivalent to the integral equation

$$x(t) - \int_{-\infty}^{\infty} G_0(t,s) F[\varepsilon, s, x(s)]\, ds = 0, \tag{15}$$

where

$$F(\varepsilon, s, x) = \begin{cases} X(s/\varepsilon, x) - Bx, & \text{if } \varepsilon \ne 0, \\ 0, & \text{if } \varepsilon = 0. \end{cases}$$

The left-hand side of equation (15) is an operator $f(\varepsilon, x)$ acting in the space $\mathcal{C}$. It is not difficult to see that, by virtue of (10), this operator is Fréchet differentiable at the zero point, and

$$f'_x(\varepsilon, 0)h = h(t) - \int_{-\infty}^{\infty} G_0(t,s) F'_x(\varepsilon, s, 0) h(s)\, ds.$$

In this case condition (1) is satisfied. Now, to prove the theorem, one may use Theorem 1.

Condition a) of Theorem 1 follows from (11) and the lemma.

The proof of condition b) rests on (12) and Theorem 2. Indeed, from the equality $f'_x(\varepsilon, 0)h = g$ it follows that the function $h - g$ is a bounded solution of the differential equation

$$dz/dt = Bz + F'_x(\varepsilon, t, 0)h(t)$$

or, what is the same, the equation

$$dz/dt = A(t/\varepsilon)z + [A(t/\varepsilon) - B]g(t).$$

By virtue of Theorem 2,

$$\|h - g\|_{\mathcal{E}} \le \|G_\varepsilon\| L \|g\|_{\mathcal{E}},$$

where

$$L = \sup_{-\infty<t<\infty} |A(t) - B|,$$

whence

$$\|h\|_{\mathcal{E}} \leq c\|g\|_{\mathcal{E}}.$$

In other words, all the operators $f'_x(\varepsilon, 0)$ have uniformly bounded inverse operators

$$\Gamma(\varepsilon) = [f'_x(\varepsilon, 0)]^{-1}.$$

Condition b) of Theorem 1 is fulfilled.

4. Let us make a few remarks. Above, for differential equations with a small parameter, the problem of bounded solutions was considered. If the right-hand sides of equation (13) are uniformly, with respect to x, almost periodic in t (or ω-periodic in t), then it is easy to show that the existing unique bounded solution $x_\varepsilon(t)$ is almost periodic (ω-periodic).

Theorem 4 carries over to equations in Banach spaces.

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Note: Figure translations are in progress. See original paper for figures.