ON CANONICAL DECOMPOSITIONS OF EQUATIONS OF A LOCAL ANALYTIC QUASIGROUP

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Abstract

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ON CANONICAL DECOMPOSITIONS OF EQUATIONS OF A LOCAL ANALYTIC QUASIGROUP

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1. Let X, Y, Z be three analytic manifolds of the same dimension r. An analytic mapping $f : X \times Y \to Z$ is called a local analytic quasigroup Q_r if, for $a \in X$, $b \in Y$, and $c = f(a, b) \in Z$, the following conditions are satisfied:

a) For every neighborhood U_c of the point c there exist neighborhoods U_a and U_b of the points a and b such that, for any $x \in U_a$, $y \in U_b$, the function $f(x, y)$ is defined and $f(x, y) \in U_c$.

b) For every neighborhood U_a of the point a there exist neighborhoods U_b and U_c of the points b and c such that, for any $y \in U_b$ and $z \in U_c$, the equation $f(x, y) = z$ has a unique solution with respect to x, and $x \in U_a$.

c) For every neighborhood U_b of the point b there exist neighborhoods U_a and U_c of the points a and c such that, for any $x \in U_a$ and $z \in U_c$, the equation $f(x, y) = z$ has a unique solution with respect to y, and $y \in U_b$.

If the manifolds X, Y, Z coincide, $X \equiv Y \equiv Z$, and there exists an element e such that $f(x, e) = f(e, x) = x$, then the quasigroup Q_r becomes a local analytic loop ([1]). If, moreover, the associativity condition $f[x, f(y, z)] = f[f(x, y), z]$ is satisfied on the quasigroup Q_r, then it becomes a local r-parameter Lie group.

In the present paper canonical decompositions are obtained for the analytic function $z = f(x, y)$ defining the quasigroup Q_r, which pass into the Campbell-Hausdorff formula ([2]) when the quasigroup Q_r becomes a local Lie group. The constructed canonical decompositions are applied to the study of certain special classes of local analytic quasigroups.

2. Let a and b be arbitrary points of the manifolds X and Y, and let $c = f(a, b)$ be a point belonging to the manifold Z. Introduce local coordinates in neighborhoods U_a, U_b, and U_c of the points a, b, and c. Then in these neighborhoods the mapping $f : X \times Y \to Z$ is written in the form:

$$z^i = f^i(x^j, y^k), \quad i, j, k = 1, \dots, r, \tag{1}$$

where x^i are the coordinates of the point $x \in U_a$, y^i are the coordinates of the point $y \in U_b$, z^i are the coordinates of the point $z \in U_c$, and f^i are analytic functions.

Suppose that the points a and b of the manifolds X and Y are determined by the coordinates $x^i = 0$ and $y^i = 0$. Then in the neighborhoods U_a and U_b the right-hand sides of equations (1) can be represented in the form of convergent power series:

$$z^i = \sum_{s=0}^{\infty} \underset{(s)}{\Lambda^i}(x^j, y^k), \tag{2}$$

where

$$\underset{(s)}{\Lambda^i}(x^j, y^k) = \frac{1}{s!}\sum_{p=0}^{s} C_s^p \lambda^i_{j_1 \dots j_p\, r+1\, j_{p+1} \dots r+j_s} x^{j_1} \dots x^{j_p} y^{j_{p+1}} \dots y^{j_s}. \tag{3}$$

homogeneous polynomials of degree s in x^j, y^k. Their coefficients $\lambda^i_{j_1 \dots j_p\, r+j_{p+1} \dots r+j_s}$ are symmetric in the indices $j_1 \dots j_p$ and $j_{p+1} \dots j_s$, and C_s^p are binomial coefficients.

The local coordinates of points of the neighborhoods U_a, U_b, and U_c of the analytic manifolds X, Y, Z admit transformations of the form

$$\tilde{x}^i = \varphi^i(x^j), \qquad \tilde{y}^i = \psi^i(y^j), \qquad \tilde{z}^i = \chi^i(z^j), \tag{4}$$

where $\varphi^i, \psi^i, \chi^i$ are uniquely invertible analytic functions such that $\varphi^i(0) = 0$, $\psi^i(0) = 0$. Such transformations are called isotopic coordinate transformations of the quasigroup Q_r. These transformations make it possible to reduce the expansions (2) to a certain simplest form, namely:

Theorem 1. *With the aid of transformations (4), the polynomials $\Lambda^i_{(s)}(x^j, y^k)$ occurring in the expansions (2) can be reduced to the following canonical form:*

$$\Lambda^i_{(0)} = 0, \qquad \Lambda^i_{(1)} = x^i + y^i, \qquad \Lambda^i_{(2)} = a^i_{jk} x^j y^k,$$

$$\Lambda^i_{(s)} = \frac{1}{s!}\sum_{p=1}^{s-1} C_s^p a^i_{j_1 \dots j_p\, r+j_{p+1} \dots r+j_s} x^{j_1} \dots x^{j_p} y^{j_{p+1}} \dots y^{j_s}, \tag{5}$$

where the coefficients of these polynomials satisfy the relations

$$a^i_{(jk)} = 0, \qquad b^i_{(j_1 \dots j_s)} = 0, \tag{6}$$

where

$$b^i_{(j_1\dots j_s)} = \sum_{p=1}^{s-1} C^p_s \sigma^{s-p} a^i_{j_1\dots j_p\, r+j_{p+1}\dots r+j_s},$$

and σ is some fixed real number, $\sigma \neq 0, -1$.

Expansions of equations (2) satisfying the conditions of this theorem will be called canonical expansions of the equations of the quasigroup Q_r. Thus, the quasigroup Q_r has not one but an entire pencil of canonical expansions, depending on the parameter σ. The variables x^i and y^i occurring in the canonical expansions of the quasigroup Q_r will be called its canonical parameters.

The uniqueness of the canonical expansions is confirmed by the following theorem:

Theorem 2. *The canonical expansion of the equations of a local analytic quasigroup preserves its form if and only if the variables entering into it are transformed according to the formulas*

$$\tilde{x}^i = \alpha^i_j x^j, \qquad \tilde{y}^i = \alpha^i_j y^j, \qquad \tilde{z}^i = \alpha^i_j z^j. \tag{7}$$

In this case the coefficients a^i_{jk} and $a^i_{j_1\dots j_p\, r+j_{p+1}\dots r+j_s}$ of the canonical expansion are transformed according to the tensor law.

The proof of Theorems 1 and 2 is carried out by the method of complete induction on the degree s of the polynomials $\Lambda_{(s)}(x, y)$.

3. Let us identify the points of the neighborhoods U_a, U_b, and U_c that have identical coordinates, and denote by the letter e the point obtained by identifying the points a, b, c. Then the local quasigroup Q_r will be defined in some neighborhood U of this point, and the binary quasigroup operation will be an analytic mapping $f : U \times U \to U$. In this case the point e will correspond to the identity element of the quasigroup Q_r, since by virtue of relations (5)

$$f(x, e) = f(e, x) = x.$$

Consequently, the quasigroup Q_r becomes a local analytic loop. Moreover, by virtue of relations (5) and (6),

$$f(x, \sigma x) = (1 + \sigma)x, \tag{8}$$

where the latter relation is satisfied for one fixed value of the parameter $\sigma \neq 0, -1$. We shall call condition (8) the normalization condition for the equations of the quasigroup.

4. Let Q_1 be a one-parameter quasigroup. By virtue of the first of conditions (6), its canonical decompositions are written in the form

$$z = x + y + \sum_{s=3}^{\infty} \underset{(s)}{\Lambda}(x, y).$$

If we introduce the notation

$$a_{\underbrace{1\ldots1}_{p}\;\underbrace{2\ldots2}_{q}} = a_{pq},$$

then the polynomials $\underset{(s)}{\Lambda}(x, y)$, for $s \geqslant 3$, take the form

$$\underset{(s)}{\Lambda}(x, y) = \frac{1}{s!}\sum_{p=1}^{s-1} C_s^p a_{p,s-p} x^p y^{s-p}, \tag{9}$$

where

$$\sum_{p=1}^{s-1} C_s^p \sigma^{s-p} a_{p,s-p} = 0.$$

A direct verification shows that if the quasigroup Q_1 becomes a one-parameter Lie group G_1, then all its canonical decompositions take the form

$$z = x + y.$$

Consequently, if the quasigroup Q_r becomes a Lie group, then the canonical parameters introduced on Q_r pass into the ordinary canonical parameters of the local Lie group G_r. All canonical decompositions of the quasigroup Q_r then pass into the Campbell–Hausdorff formula for the Lie group G_r.

5. Consider on the quasigroup Q_r a one-parameter subquasigroup Q_1. It is defined on Q_r by the equations:

$$x^i = x^i(u), \qquad y^i = y^i(v), \qquad f^i[x^j(u), y^h(v)] = z^i(w), \tag{10}$$

where $w = \varphi(u, v)$. From these relations the following theorem follows easily:

Theorem 3. In order that the first two of equations (10) define on the quasigroup Q_r a one-parameter subquasigroup Q_1, it is necessary and sufficient that the relations

$$\lambda\frac{\partial z^i}{\partial x^j}\frac{dx^j}{du} + \mu\frac{\partial z^i}{\partial y^j}\frac{dy^j}{dv} = 0, \tag{11}$$

hold, where λ and μ are analytic functions of u and v.

Relations (11) are differential equations of one-parameter subquasigroups on Q_r. These relations make it possible to prove the following theorem:

Theorem 4. In order that on the loop Q_r, for each direction issuing from the point e, there exist a one-parameter subloop Q_1 tangent to this direction, it is necessary and sufficient that the coefficients of at least one of its canonical decompositions satisfy the relations:

$$\underset{p}{b}^i_{(j_1\cdots j_s)} = \delta^i_{(j_1}\underset{p}{b}_{j_2\cdots j_s)}, \tag{12}$$

where

$$\underset{p}{b}^i_{j_1\cdots j_s} = a^i_{j_1\cdots j_p\, r+j_{p+1}\cdots r+j_s}.$$

We note that if the condition of Theorem 4 is fulfilled, the coefficients of any of the canonical decompositions of the quasigroup Q_r will satisfy relations (12).

The three-web ([3]) corresponding to the quasigroup Q_r defined by Theorem 4 was considered by us in ([4]) under the name of a transversal-geodesic web.

6. The following theorem answers the question of when the one-parameter subloops Q_1 of the loop Q_r become groups.

Theorem 5. *In order that on the loop Q_r, for every direction issuing from the point e, there exist a one-parameter subgroup G_1 tangent to this direction, it is necessary and sufficient that the coefficients of at least one of its canonical decompositions satisfy the conditions*

$$\underset{p}{b}^i_{(j_1\cdots j_s)} = 0. \tag{13}$$

The condition contained in Theorem 5 is equivalent to the monoassociativity ([3]) of the loop Q_r. Therefore condition (13) is a necessary and sufficient condition for the monoassociativity of the loop Q_r. Note that in ([1]) the existence of one-parameter subgroups of a local loop Q_r was proved only for alternative loops, which form a narrower class than monoassociative loops.

When condition (13) is fulfilled, for arbitrary x belonging to a sufficiently small neighborhood of the point e of the loop Q_r, and for sufficiently small real λ and μ, the relation

$$f(\lambda x, \mu x) = (\lambda + \mu)x$$

will hold, and this means that the normalization condition (8) is fulfilled on the quasigroup Q_r for every $\sigma \neq 0, -1$. Hence it follows:

Theorem 6. *In order that all the canonical decompositions of the equations of the quasigroup Q_r introduced in Theorem 1 coincide with one another, it is necessary and sufficient that this quasigroup be monoassociative.*

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Note: Figure translations are in progress. See original paper for figures.