NECESSARY CONDITIONS FOR A MINIMUM FOR ONE TYPE OF NONSMOOTH PROBLEMS

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Abstract

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NECESSARY CONDITIONS FOR A MINIMUM FOR ONE TYPE OF NONSMOOTH PROBLEMS

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The note considers a minimization problem with a nonsmooth equality-type constraint. To investigate the problem, one of the schemes of the method of feasible directions is applied: the derivative on the cone of feasible directions is nonnegative (one of the schemes was developed in ([1])). The main difficulty of the investigation consists in describing the cone of feasible directions or the directional derivative. In particular, some minimax problems, problems of minimizing a function on the set of extrema of another function, are included in our problem, and for them necessary conditions for a minimum are written out.

Problem (1)–(4). Let real functions $\varphi(y,x)$, $f_\alpha(y,x)$, $\alpha \in A$, $y \in Y$, $x \in X$, and an operator $H : Y \times X \to Z$ be given, where A is some set of indices; Y, X, Z are complete normed spaces. For every $y \in Y$ for which the set

$$\mathfrak{M}(y) = \{x \mid H(y,x) = 0,\ \max_{\alpha \in A} f_\alpha(y,x) \leq 0\} \neq \emptyset$$

is given, define uniquely the operator

$$P : y \to P(y) \in \mathfrak{M}(y).$$

It is required to find

$$\min_{(y,x)} \varphi(y,x) \tag{1}$$

on the set of pairs (y,x) satisfying the constraints:

$$H(y,x) = 0, \tag{2}$$

$$\max_{\alpha \in A} f_\alpha(y, x) \le 0, \tag{3}$$

$$x = P(y). \tag{4}$$

We shall agree to denote by (y^*, x^*) a minimum point in problem (1)–(4). With respect to the operator H and the functions f_α, everywhere in what follows we assume the following: the operator $H(y, x)$ is continuously differentiable with respect to y, x, and H'_x maps X onto all of Z at each point (y, x), $x \in \mathfrak{M}(y)$; the functions $f_\alpha(y, x)$ are convex jointly in y, x and satisfy a Lipschitz condition with a uniformly bounded Lipschitz constant on every bounded domain in $Y \times X$; there exists a vector $(\bar{y}', \bar{x}') \in Y \times X$ such that

$$H'_y(y^*, x^*) + H'_x(y^*, x^*)\bar{x}' = 0, \qquad \max_{\alpha \in A} f_\alpha(y^* + \bar{y}', x^* + \bar{x}') < 0.$$

Introduce the notation:

$$\mathcal{K} = \{(\bar{y}, \bar{x}) \mid \exists \lambda' > 0,\ \max_{\alpha \in A} f_\alpha(y^* + \lambda' \bar{y},\ x^* + \lambda' \bar{x}) < 0\},$$

$$L = \{(\bar{y}, \bar{x}) \mid H'_y(y^*, x^*)\bar{y} + H'_x(y^*, x^*)\bar{x} = 0\},$$

$$K = L \cap \mathcal{K}, \qquad K(\bar{y}) = \{\bar{x} \mid (\bar{y}, \bar{x}) \in K\},$$

$$\overline{K}(\bar{y}) = \{\bar{x} \mid (\bar{y}, \bar{x}) \in \overline{K}\}.$$

$\mathrm{Pr}_Y K$ is the projection of the cone K onto the space Y.

Assertion 1. Suppose that for any sequence y_ε of the form $y_\varepsilon = y^* + \varepsilon\bar{y} + o(\varepsilon\bar{y})$, $\bar{y} \in \mathrm{Pr}_Y K$, $\varepsilon \ge 0$, for which $\mathfrak{M}(y_\varepsilon) \ne \emptyset$, there exists a finite or infinite limit

$$\lim_{\varepsilon \to +0} \frac{\varphi(y_\varepsilon, P(y_\varepsilon)) - \varphi(y^*, x^*)}{\varepsilon} = \frac{\partial \varphi(y^*, P(y^*))}{\partial \bar{y}},$$

called the directional derivative in the direction $\bar{y}$ of the function $\varphi(y, P(y))$ at the point (y^*, x^*).

In order that the point (y^*, x^*) furnish a minimum in problem (1)–(4), it is necessary that

$$\inf_{\bar{y} \in \mathrm{Pr}_Y K} \frac{\partial \varphi(y^*, P(y^*))}{\partial \bar{y}} \ge 0. \tag{5}$$

The principal difficulty lies in establishing the existence and finding an explicit form of $\partial\varphi(y^*, P(y^*))/\partial\bar{y}$. Here these questions are considered for the operator $P : y \to P(y) \in \mathfrak{M}(y)$, where $P(y)$ is the solution of a certain maximization problem:

$$\psi(y, P(y)) = \max_{x \in \mathfrak{M}(y)} \psi(y, x).$$

The minimax problem considered here is included in problem (1)–(4) when $\varphi(y, x) \equiv \psi(y, x)$. It is an essential complication and generalization of the usual minimax problem of the form

$$\min_{y \in Q_y} \max_{x \in Q_x} \psi(y, x), \qquad Q_y \subset Y, \quad Q_x \subset X,$$

since both variables y, x are connected by relations (2), (3).

Problem (6). Find a point (y^*, x^*) furnishing a minimax of $\psi(y, x)$, i.e.

$$\psi(y^*, x^*) = \max_{x \in \mathfrak{M}(y^*)} \psi(y^*, x^*) = \min_{y : \mathfrak{M}(y) \neq \emptyset} \max_{x \in \mathfrak{M}(y)} \psi(y, x). \tag{6}$$

A necessary condition for a minimax of $\psi(y, x)$ at the point (y^*, x^*) is given by the following

Theorem 1. Let the operator $H(y, x)$ be linear in x for each y; if $K(0) = \emptyset$, then $\overline{K}(0) = \{0\}$, and the Hausdorff distance $\rho(\overline{K}(\bar{y}), \overline{K}(0)) \to 0$ as $\|\bar{y}\| \to 0$, $\bar{y} \in \operatorname{Pr}_Y K$. The function $\psi(y, x)$ is continuously differentiable in y, x, concave in x, attains its unique maximum on $\mathfrak{M}(y)$, and the maximization problem for $\psi(y, x)$ on $\mathfrak{M}(y)$ is stable with respect to the solution (i.e. from $y \to y^*$ it follows that $P(y) \to P(y^*)$, if $\mathfrak{M}(y) \neq \emptyset$).

Then, in order that the point (y^*, x^*) furnish a minimax of the function $\psi(y, x)$, it is necessary that for any linear functionals $w_1 = (y_1^*, x_1^*) \in \mathcal{K}^*$, $w_2 = (y_2^*, x_2^*) \in L^*$, for which

$$\psi'_x(y^*, x^*) + x_1^* + x_2^* = 0, \tag{7}$$

the relation

$$(\psi'_y(y^*, x^*) + y_1^* + y_2^*, 0) \in \mathcal{K}^* + L^* \tag{8}$$

hold; moreover, if $K(0) \neq \emptyset$, then (8) is equivalent to

$$\psi'_y(y^*, x^*) + y_1^* + y_2^* = 0.$$

The set of linear functionals $w_1 \in \mathcal{K}^*$, $w_2 \in L^*$, satisfying (7), is nonempty.

The conditions of the theorem ensure the differentiability of the function

$$\psi(y, P(y)) = \max_{x \in \mathfrak{M}(y)} \psi(y, x)$$

in each direction $\bar{y} \in \operatorname{Pr}_Y K$ at the point

(y^*, x^*) and $\partial\psi(y^*, P(y^*))/\partial\bar{y} = \sup_{\bar{x} \in K} \psi'(y^*, x^*)\bar{y}\bar{x}$. Hence it is not difficult to obtain that

$$-\inf_{\bar{y} \in \operatorname{Pr} YK} \sup_{\bar{x} \in K(\bar{y})} \psi'(y^*, x^*)\bar{y}\bar{x} = 0. \tag{9}$$

Condition (9) is equivalent to the assertion of the theorem (7), (8). In proving this fact the following fact is used.

Let in the normed space Y a convex cone Ω with an interior point, with vertex at O, be given, and let on it a linearly concave functional $f(\bar{y})$ be defined, i.e.

$$f(\alpha\bar{y}) = \alpha f(\bar{y}), \quad \alpha > 0, \quad \bar{y} \in \Omega,$$
$$f(\bar{y}_1 + \bar{y}_2) \geq f(\bar{y}_1) + f(\bar{y}_2), \quad \bar{y}, \bar{y}_2 \in \Omega,$$
$$|f(\bar{y})| \leq C\|\bar{y}\|, \quad \bar{y} \in \Omega;$$

then for any $\bar{y}^0 \in \Omega^0$ there exists a supporting functional $l \in Y^*$: $l(\bar{y}) \geq f(\bar{y})$, $\bar{y} \in \Omega$, such that $l(\bar{y}^0) = f(\bar{y}^0)$.

Remark 1. The condition of concavity of $\psi(y, x)$ with respect to x may be omitted when $K(0) = \{0\}$.

Let us note that the necessary condition formulated, as a first-order condition, is only a stationarity condition and does not separate a minimum-maximum from a maximum-maximum.

Remark 2. If there exists a vector $(\bar{y}, \bar{x}) \in K$, $\psi'(y^*, x^*)\bar{y}\bar{x} > 0$, then the necessary minimax condition may be written in the form

$$-\psi'_x(y^*, x^*) \in \overline{K}(0)^*,$$

$$\operatorname{Pr}_Y K \subseteq \operatorname{Pr}_Y\{(\bar{y}\bar{x}) \mid \psi'(y^*, x^*)\bar{y}\bar{x} \geq 0\} \cap K\}.$$

Theorem 2. Let the functions $f_\alpha(y, x)$, $\alpha \in A$, the operator H, and the function $\psi(y, x)$ be as in Theorem 1, and let $\varphi(y, x)$ be continuously differentiable with respect to y, x. Suppose, in addition, that the following conditions are satisfied:

a) $|\varphi(y,x)-\varphi(y,x')| \le k_1|\psi(y,x)-\psi(y,x')|$, $0<k_1<\infty$, for $x, x' \in \mathfrak{M}(y)$;

b) the problem $\sup\limits_{\bar{x}\in\overline{K}(\bar{y})} \psi'(y^*,x^*)\bar{y}\bar{x}$ has a solution for every $\bar{y} \in \operatorname{Pr}_Y K$.

Then, in order that the point (y^*,x^*) deliver a minimum in problem (1)–(4), it is necessary that

$$\inf_{(\bar{y},\bar{x})\in\Gamma} \varphi'(y^*,x^*)\bar{y}\bar{x} = 0,$$

where

$$\Gamma = \{(\bar{y}\bar{x}) \mid \psi'(y^*,x^*)\bar{y}\bar{x} = \sup_{\bar{x}'\in\overline{K}(\bar{y})} \psi'(y^*,x^*)\bar{y}\bar{x}',\ \bar{y}\in \operatorname{Pr}_Y K\}.$$

Example. Find $\min \varphi(y,x)$ under the constraints $y + a_1x \le 0$, $-y - a_2x \le 0$, $x = P(y)$, where $P(y)$:

$$\varphi(y,P(y)) = \max_{x\in\mathfrak{M}(y)} \varphi(y,x), \quad \mathfrak{M}(y) = \{x \mid y + a_1x \le 0,\ -y - a_2x \le 0\}.$$

Assume that the point $(0,0)$ gives a minimum of $\varphi(y,x)$, and write the necessary conditions for a minimum. $K^* = \{\lambda_1(-1,-a_1) + \lambda_2(1,a_2),\ \lambda_1,\lambda_2 \ge 0\}$.

According to Theorem 1, for any linear functional $w_1 \in K^*$ for which

$$\varphi'_x(0,0) - \lambda_1a_1 + \lambda_2a_2 = 0,$$

there exists $w_2 \in K^*$, $w_2 = \lambda_3(-1,-a_1) + \lambda_4(1,a_2)$, $\lambda_1,\lambda_2 \ge 0$, such that

$$\varphi'_y(0,0) - \lambda_1 + \lambda_2 = -\lambda_3 + \lambda_4, \quad -\lambda_3a_1 + \lambda_4a_2 = 0.$$

Let $a_2 \ne 0$ and $0 < a_1/a_2 \le 1$. Then $\varphi'_y(0,0) - \lambda_1 + \lambda_2 = \lambda_3(-1 + a_1/a_2) \le 0$ for all $\lambda_2 \ge 0$, $\lambda_2 = [-\varphi'_x(0,0) + \lambda_1a_1]/a_2$, where $\lambda_1 \ge 0$.

The latter is either solvable for any $\lambda_1 \geqslant 0$, and then $\varphi'_y(0,0) - \varphi'_x(0,0)/a_2 \leqslant 0$, or solvable for $\lambda_2 = 0$, $\lambda_1 \geqslant 0$, and then $\varphi'_y(0,0) - \varphi'_x(0,0)/a_1 \leqslant 0$.

If $a_1a_2 < 0$, then $\lambda_3 = \lambda_4 = 0$, and, consequently,

$$\varphi'_y(0,0) - \lambda_1 + \lambda_2 = 0$$

for all those $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$ for which

$$\varphi'_x(0,0) - \lambda_1a_1 + \lambda_2a_2 = 0.$$

It follows easily from this that $\varphi'_y(0,0) = \varphi'_x(0,0) = 0$. If $a_1 = 0,\ a_2 \neq 0$, we obtain that

$$\operatorname{sgn}\varphi'_x(0,0) = -\operatorname{sgn} a_2, \qquad \varphi'_y(0,0) - \frac{1}{a_2}\varphi'_x(0,0) \leqslant 0.$$

Thus, a necessary condition for a minimum of $\varphi(y,x)$ at the point $(0,0)$ is

$$\varphi'_y(0,0) - \frac{1}{a_2}\varphi'_x(0,0) \leqslant 0$$

or

$$\varphi'_y(0,0) - \frac{1}{a_1}\varphi'_x(0,0) \leqslant 0$$

for $0 < a_1/a_2 \leqslant 1$, and equality when $a_1 = a_2$;

$$\varphi'_y(0,0) - \frac{1}{a_2}\varphi'_x(0,0) \leqslant 0$$

and

$$\operatorname{sgn}\varphi'_x(0,0) = -\operatorname{sgn} a_2$$

for $a_1 = 0,\ a_2 \neq 0$;

$$\varphi'_y(0,0) = \varphi'_x(0,0) = 0$$

for $a_1 a_2 < 0$.

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Note: Figure translations are in progress. See original paper for figures.