ON SINGULAR INTEGRAL TRANSFORMATIONS IN WEIGHTED L_p-SPACES

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Abstract

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ON SINGULAR INTEGRAL TRANSFORMATIONS IN WEIGHTED L_p-SPACES

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The well-known theorem of Riesz ([1]) on the Hilbert transform of functions $u(x) \in L_p(-\infty, \infty)$, $1 < p < \infty$, admits generalizations in various directions. K. I. Babenko ([2]) (see also ([3])) showed that the analogous theorem remains valid also for functions satisfying the condition

$$\int_{-\infty}^{\infty} |x|^{\beta} |u(x)|^p dx < \infty \qquad (1 < p < \infty) \tag{1}$$

for all values of β for which $-1/p < \beta < 1/p'$, $1/p + 1/p' = 1$. Calderon and Zygmund ([4]), considering in the n-dimensional Euclidean space R_n of points $x(x_1, \dots, x_n)$ singular integrals of the form

$$v(x) = \int_{R_n} \frac{f(x, (x-y)/|x-y|)}{|x-y|^n} u(y)\, dy \qquad \left(|x| = \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2} \right), \tag{2}$$

under certain conditions on the characteristic $f(x, (x-y)/|x-y|)$, proved a theorem analogous to Riesz' s theorem for functions $u(x) = u(x_1, \dots, x_n)$ of many variables. Generalizing this theorem and the result of K. I. Babenko, Stein ([5]) showed that if $f(x, (x-y)/|x-y|)$ is bounded and $1 < p < \infty$, then the inequality

$$\int_{R_n} ||x|^{\beta} v(x)|^p dx \leqslant A_{p,\beta} \int_{R_n} ||x|^{\beta} u(x)|^p dx, \tag{3}$$

holds as soon as (1) is satisfied, for all values of β for which

$$-n/p < \beta < n/p'. \tag{4}$$

From the results of Calderon–Zygmund and Stein it follows that, under the same restrictions on β and p, for all functions $u(x)$ for which

$$\int_{R_n} |u(x)(1+|x|)^{\beta}|^p dx < \infty,$$

the inequality

$$\int_{R_n} |v(x)(1+|x|)^{\beta}|^p dx \leqslant A_{p,\beta} \int_{R_n} |u(x)(1+|x|)^{\beta}|^p dx$$

is valid, where $A_{p,\beta}$ is a constant independent of $u(x)$. S. G. Mikhlin ([6], p. 63) posed the question of the validity of this inequality for $p=\infty$ under condition (4), i.e., when $0<\beta<n$. In the author' s paper ([7]) a negative answer to this question is given. Moreover, it can be proved that there does not exist a function $\lambda(t)$, monotonically decreasing to zero on the half-axis $0<t<\infty$, such that for all $u(x)$ satisfying the condition $|u(x)| \leqslant A\lambda(|x|)$, the inequality $|v(x)| \leqslant B\lambda(|x|)$ is valid.

Developing a method going back to Hardy and Littlewood ([3]), it is possible to generalize Stein' s result and obtain analogues of (3) for typically multidimensional–

inequalities in which, instead of the weight $|x|^{\beta}$, one considers weights of the form
$\left(\sum_{\nu=1}^{k} |x_\nu|\right)^{\beta}$ and $\prod_{\nu=1}^{k} |x_\nu|^{\beta_\nu}$ for an arbitrary natural $k \leq n$, as well as more general weights. In passing from the usual classes L_p to weighted classes L_p, in order that the integral (2) exist it becomes necessary to impose certain restrictions on the weights under consideration. These conditions, together with the conditions on the weights under which the Calderón–Zygmund operator (2) is bounded in the corresponding weighted spaces, under the assumption that
$\sup\left|f\left(x, \frac{x-y}{|x-y|}\right)\right| < \infty$, are given in Table 1.

Table 1

	1	2	3	4	5	6	7
I	Weights	$\|x\|^{\beta}$	$\left(\sum_{\nu=1}^{k} \|x_\nu\|\right)^{\beta}$	$\sum_{\nu=1}^{k} \|x_\nu\|^{\beta}$	$\prod_{\nu=1}^{k} \|x_\nu\|^{\beta_\nu}$	$(1+\|x\|)^{\beta}$	$\left(\frac{\|x\|}{1+\|x\|}\right)^{\beta}$

	1	2	3	4	5	6	7
II	Conditions for convergence of the integral (2)	$-\frac{n}{p} < \beta < \frac{n}{p'}$	$-\frac{n}{p} < \beta < \frac{k}{p'}$	$-\frac{n}{p} < \beta < \frac{k}{p'}$	$-\frac{1}{p} < \beta_\nu < \frac{1}{p'}$, $\nu = 1, \dots, k$	$-\frac{n}{p} < \beta$	$\beta < \frac{n}{p'}$
III	Conditions for preservation of the weight class	$-\frac{n}{p} < \beta < \frac{n}{p'}$	$-\frac{k}{p} < \beta < \frac{k}{p'}$	$-\frac{k}{p} < \beta < \frac{1}{p'}$	$-\frac{1}{p} < \beta_\nu < \frac{1}{p'}$, $\nu = 1, \dots, k$	$-\frac{n}{p} < \beta < \frac{n}{p'}$	$-\frac{n}{p} < \beta < \frac{n}{p'}$

The restrictions on the exponents β given in row II of this table are not only sufficient but, with the exception of the left inequality (II, 5), also necessary in order that all integrals (2) converge on the corresponding weighted class. The left inequality (II, 5) could be weakened by replacing it, for example, by the condition $\beta_1 > (-n+k-1)/p$, $\beta_\nu > -1/p$ $(\nu = 2, \dots, n)$. It is noteworthy that the conditions (III, 2)–(III, 7) are more restrictive than the corresponding conditions (II, 2)–(II, 7). However, all the indicated bounds for β in row III of the table are unimprovable. They are necessary and sufficient for the invariance of the corresponding weighted class with respect to Calderón–Zygmund transformations. The results (III, 3) and (III, 5) are contained in the following more general theorem.

Theorem 1. Let $1 = l_0 < l_1 < \dots < l_k \le n$,

$$-\frac{l_\nu - l_{\nu-1}}{p} < \beta_\nu < \frac{l_\nu - l_{\nu-1}}{p'}, \qquad 1 < p < \infty, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

and suppose that the function $u(x)$ has the property that

$$\int_{R_n} \left| \prod_{\nu=1}^{k} \left(\sum_{i=l_{\nu-1}}^{l_\nu - 1} |x_i| \right)^{\beta_\nu} u(x) \right|^p dx < \infty.$$

Then, for $v(x)$ defined by the operator (2), the inequality

$$\left\{ \int_{R_n} \left| \prod_{\nu=1}^{k} \left(\sum_{i=l_{\nu-1}}^{l_\nu - 1} |x_i| \right)^{\beta_\nu} v(x) \right|^p dx \right\}^{1/p} \le$$

$$\leq A_{p,\beta_1,\dots,\beta_k}\left\{\int_{R_n}\left|\prod_{\nu=1}^{k}\left(\sum_{i=l_{\nu-1}}^{l_\nu-1}|x_i|\right)^{\beta_\nu}u(x)\right|^p dx\right\}^{1/p},$$

where $A_{p,\beta_1,\dots,\beta_k}$ is a constant independent of $u(x)$. The ranges of variation for β_ν $(\nu=1,\dots,k)$ are sharp.

The results noted remain valid if, instead of the whole space R_n, one considers any part of it and, in particular, the half-space $R_n^+\{(x_1,\dots,x_{n-1})\in R_{n-1}, x_n>0\}$. Along with the consideration on R_n^+ of transformation (2), of known interest (see, for example, ([8])) is the study on this half-space of transformations of the form

$$v(x)=\int_{R_n^+}\frac{f(x,y/|y|)}{|y|^n}\,u(y+x)\,dy$$

under additional restrictions on the characteristic $f(x,y/|y|)$. For such transformations S. V. Uspenskii, in particular, proved that if $\beta\geq 0$ and

$$\int_{R_n^+}|x_n^\beta u(x)|^p\,dx<\infty,$$

then $v(x)$ has the same property and the inequality holds

$$\left\{\int_{R_n^+}|x_n^\beta v(x)|^p\,dx\right\}^{1/p}\leq c\left\{\int_{R_n^+}|x_n^\beta u(x)|^p\,dx\right\}^{1/p}. \tag{5}$$

The method we use makes it possible, relying on inequality (5) for $\beta=0$, to weaken the restriction $\beta\geq 0$ in S. V. Uspenskii' s assertion, replacing it by the restriction $\beta>-1/p$. This method also makes it possible to strengthen the general theorem of S. V. Uspenskii for the half-space, which is related to the generalization, considered earlier by O. V. Besov and P. I. Lizorkin ([9]), of the Calderón–Zygmund transformation of the form

$$v(x)=\int K(y)\,u(x+y)\,dy$$

with kernel $K(x)$ possessing the property of "generalized" homogeneity (see ([9])). Thanks to this it is also possible to strengthen the estimates obtained by S. V. Uspenskii for weighted norms of mixed derivatives of differentiable functions of many variables. In particular, the following is valid.

Theorem 2. Let G be a bounded domain with smooth boundary $\partial G\in C^{(l)}$, and let

$$\|f\|_{W^{l,\dots,l}_{p,\alpha,\dots,\alpha}(G)} = \|f\|_{W^{l}_{p,\alpha}(G)} = \left\{ \|f\|_{L_p(G)} + \sum_{i=1}^{n} \left\| \rho^{\alpha/p} \frac{\partial^{l_i} f}{\partial x_i^{l_i}} \right\|_{L_p(G)} \right\} < \infty,$$

where ρ is the distance from the point x to the boundary of the domain G. Then, if $\varkappa = l - \sum_{i=1}^{r} k_i \geq 0$ and $\beta = \alpha/p - \varkappa > -1/p$, then

$$\left\| \rho^{\beta} \frac{\partial^{k_1+\dots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right\|_{L_p(G)} \leq C \|f\|_{W^{l}_{p,\alpha}(G)}. \tag{6}$$

This theorem for $\beta \geq 0$ was proved by S. V. Uspenskii ([8]), who, in the case when $\beta < 0$, obtained the inequality

$$\left\| \frac{\partial^{k_1+\dots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right\|_{L_p(G)} \leq C \|f\|_{W^{l}_{p,\alpha}(G)}.$$

We shall also give the following proposition.

Theorem 3. Let

$$\|f\|_{W^{l_1,\dots,l_n}_{p,0}(R_n^+)} = \left\{ \|f\|_{L_p(R_n^+)} + \sum_{i=1}^{n} \left\| \frac{\partial^{l_i} f}{\partial x_i^{l_i}} \right\|_{L_p(R_n^+)} \right\} < \infty.$$

Then, if

$$0 \geqslant \beta = l_n \left(\sum_{i=1}^{n} \frac{k_i}{l_i} - 1 \right) > -\frac{1}{p},$$

then

$$\left(\frac{x_n}{1 + x_n} \right)^{\beta} D^{k_1+\dots+k_n} f \in L_p(R_n^+)$$

and

$$\left\| \left(\frac{x_n}{1 + x_n} \right)^{\beta} \frac{\partial^{k_1+\dots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right\|_{L_p(R_n^+)} \leqslant C \|f\|_{W^{l_1,\dots,l_n}_{p,0}(R_n^+)}.$$

Let us note that, under the condition

$$\sum_{i=1}^{n} \frac{k_i}{l_i} \leqslant 1,$$

L. N. Slobodetskii ([10]) obtained the inequality

$$\left\| \frac{\partial^{k_1+\dots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right\|_{L_p(R_n^+)} \leqslant C \|f\|_{W_{p,0}^{l_1,\dots,l_n}(R_n^+)}.$$

It follows from Theorem 3 that this inequality admits a strengthening in the form

$$\left\| \left(\frac{x_n}{1+x_n} \right)^{\beta} \frac{\partial^{k_1+\dots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right\|_{L_p(R_n^+)} \leqslant C \|f\|_{W_{p,0}^{l_1,\dots,l_n}(R_n^+)}. \tag{7}$$

This inequality, in contrast to the inequality of L. N. Slobodetskii, shows that when the order k of the mixed derivative is decreased within the indicated limits, the weight on its left-hand side may deteriorate. From (7), for

$$-\frac{1}{p} < l_\nu \left(\sum_{i=1}^{n} \frac{k_i}{l_i} - 1 \right) = \beta_\nu < 0$$

there follows the inequality

$$\left\| {\sum_{\nu}}' \left(\frac{x_\nu}{1+x_\nu} \right)^{\beta_\nu} \frac{\partial^{k_1+\dots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right\|_{L_p(R_n)} \leqslant C \|f\|_{W_{p,0}^{l_1,\dots,l_n}(R_n)}$$

and the inequality

$$\left\| {\prod_{\nu}}' \left(\frac{x_\nu}{1+x_\nu} \right)^{\beta_\nu} \frac{\partial^{k_1+\dots+k_n} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right\|_{L_p(R_n)} \leqslant C \|f\|_{W_{p,0}^{l_1,\dots,l_n}(R_n)},$$

where the prime on $\sum$ and $\prod$ means that only those values of ν enter the sum and the product for which the corresponding β_ν satisfy the condition $-1/p < \beta_\nu < 0$.

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Note: Figure translations are in progress. See original paper for figures.