SOLUTION OF SOME PROBLEMS ON DYADIC BICOMPACTA

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SOLUTION OF SOME PROBLEMS ON DYADIC BICOMPACTA

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Here we shall establish a certain connection between dyadicity, dimension, and metrizability, solve a number of problems posed in the papers ([1], [2]), and give answers to questions proposed to the author at P. S. Aleksandrov' s seminar at Moscow University. From the paper ([3]) we take all notation and concepts without further comment.

§ 1. **Openly dyadic bicompacta.** We shall call a bicompactum X openly dyadic if it is an open continuous image of a Tikhonov product of metric compacta

$$\prod_{\alpha\in A} R_\alpha.$$

The class $\mathfrak{R}$ of openly dyadic bicompacta is the smallest class closed with respect to Tikhonov products and open mappings and containing all metric compacta. Every finite-dimensional bicompact topological group is an openly dyadic bicompactum. Let us note a number of specific properties of openly dyadic bicompacta.

Let $X \in \mathfrak{R}$. Then:

(α). The closure of any open set U in X is of type G_δ, and the nonempty kernel of any closed set $F \subset X$ is of type F_σ. Moreover, this property is hereditary with respect to closed subsets of X of type G_δ.

(β). If X is zero-dimensional, then X is an open image of D^τ, where $\tau = wX$.

(γ). For any open mapping

$$f : \prod_{\alpha\in A} R_\alpha \to X$$

there exists an open mapping

$$g : \prod_{\alpha \in B} R_\alpha \to X,$$

where $|B| = wX$, such that $f = g\pi$, if π is the projection of

$$\prod_{\alpha \in A} R_\alpha \quad \text{onto} \quad \prod_{\alpha \in B} R_\alpha.$$

(δ). The set

$$M_{\mathfrak{n}} = \{x \in X,\ \chi(x, X) \leq \mathfrak{n}\}$$

is closed, and moreover $\chi(M_{\mathfrak{n}}, X) \leq \mathfrak{n}$ and $wM_{\mathfrak{n}} \leq \mathfrak{n}$. The set $\{x \in X,\ \chi(x, X) \geq \mathfrak{n}^+\}$ is open. The set $\{x \in X,\ \chi(x, X) = \mathfrak{n}^+\}$ is the intersection of an open set with a closed one. The set $\{x \in X,\ \chi(x, X) \geq \mathfrak{n}\}$, if $\mathfrak{n}$ is the sum of a countable number of smaller cardinals, is of type G_δ.

Lemma 1. If a bicompactum X has property (α), then the boundary of every canonical closed subset X is of type G_δ. Moreover, this property is hereditary with respect to closed subsets of X of type G_δ.

Lemma 2. Let X be a locally connected bicompactum, and let F be a closed zero-dimensional subset of type G_δ in X. Then $wF \leq \aleph_0$, and for every point $x \in F$ we have $\chi(x, X) \leq \aleph_0$*.

Theorem 1. Every finite-dimensional (in the sense of ind) locally connected openly dyadic bicompactum X is metrizable.

Proof. We shall prove that in X there exists an everywhere dense subset M with the first axiom of countability. It will follow from this that, by virtue of one theorem of the author, the metrizability of X (see (2), p. 245, or (5), pp. 162-163). Let $\operatorname{ind} X = n$. Without loss of generality we shall assume that X is connected. Otherwise X splits into a finite number of components. For every connected locally connected neighborhood $V \subset X$ there exists a neighborhood U such that $\overline{U} \subset V$ and $\operatorname{ind}(\operatorname{Fr} U) \leq n - 1$. Put $L_1 = \overline{U} \setminus \operatorname{int} \overline{U}$. Since $L_1 \subset \operatorname{Fr} U$, and by the monotonicity of the dimension ind for regular spaces ((5), p. 264), we have $\operatorname{ind} L_1 \leq n - 1$. Since X is connected, $L_1 \neq \emptyset$. Applying Lemma 1, we obtain that $\chi(L_1, X) \leq \aleph_0$. Let $\operatorname{ind} L_1 = k_1$. Clearly $k_1 \leq n - 1$. If $k_1 = 0$, then L_1 is a closed zero-dimensional subset of type G_δ lying in a locally connected bicompactum. Hence by Lemma 2 for each point $x \in L_1$ we have $\chi(x, X) \leq \aleph_0$, as required. Let $k_1 \geq 1$. We shall continue the reasoning by induction. Suppose that a nonempty closed $L_i \subset V$ of type G_δ in X has been

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* Compare with the proof of Lemma 4 in (^{4}). By $\chi(F, X)$ we denote the character of $F \subset X$, and by wX the weight of X.

found, with $0 \leq \operatorname{ind} L_i \leq n-i$. Let $\operatorname{ind} L_i = k_i$. Clearly $0 \leq k_i \leq n-i$. If $k_i = 0$, then, as we showed above, our goal has been attained. If $k_i \geq 1$, then there exists an open $W \subset L_i$ such that $\operatorname{ind}(\operatorname{Fr} W) \leq k_i - 1$. Put $L_{i+1} = \overline{W} \setminus \operatorname{int} \overline{W}$. Clearly $L_{i+1} \subset \operatorname{Fr} W \subset L_i \subset V$ and $\operatorname{ind} L_{i+1} \leq n - i - 1$. Since, by assumption, $\chi(L_i, X) \leq \aleph_0$, by Lemma 1 L_{i+1} is of type G_δ in X. The induction is complete. Thus, in no more than n steps we shall find a nonempty L_n of type G_δ in X, with $\operatorname{ind} L_n = 0$ and $L_n \subset V$. Hence by Lemma 2 there follows the existence of a point $x \in V$ such that $\chi(x, X) \leq \aleph_0$. In view of the arbitrary choice of V, we obtain the required set M. The theorem is proved.

Theorem 2. *Every connected openly dyadic peripherally metrizable bicompactum X is metrizable.**

Proof. Again we prove that in X there exists an everywhere dense subset M satisfying the first axiom of countability. Let $\mathfrak{U} = \{U\}$ be a base of open sets in X such that for every $U \in \mathfrak{U}$ we have $w(\operatorname{Fr} U) \leq \aleph_0$. Let V be an arbitrary open set in X, and let $U \in \mathfrak{U}$ be such that $\overline{U} \subset V$. Put $L = \overline{U} \setminus \operatorname{int} \overline{U}$. Then, by Lemma 1, $\chi(L, X) \leq \aleph_0$. On the other hand, $L \subseteq \operatorname{Fr} U$, and consequently $wL \leq \aleph_0$. Since X is connected, $L \neq \emptyset$. Hence for every point $x \in L$ we have

$$\chi(x, X) \leq \chi(x, L) \cdot \chi(L, X) \leq wL \cdot \chi(L, X) \leq \aleph_0 \cdot \aleph_0 = \aleph_0.$$

Since $L \subset V$, we have $x \in V$. Since V is an arbitrary open set, the set M of all points of countable character is dense in X. The theorem is proved.

Let us note that in proving Theorems 1 and 2 we used only dyadicity and property (α). Therefore the following problem naturally arises:

Is every dyadic bicompactum satisfying condition (α) openly dyadic?

Let us further note that Theorem 2 admits the following natural generalization.

Theorem 3. *If in a connected openly dyadic bicompactum X there exists a π-base $\mathfrak{U}$ such that for every $U \in \mathfrak{U}$ it follows that $w(\operatorname{Fr} U) \leq \tau$, then $wX \leq \tau$.*

§ 2. Wild dyadic bicompacta

A dyadic bicompactum X of weight τ will be called **wild** if: a) for every open $U \subset X$ we have $wU = wX = \tau$; b) for every open $U \subset X$ there exists a point $x \in U$ such that $\chi(x, X) < \tau$.

Let us note that, as follows from one theorem of the author ((2), p. 231), the weight of any wild dyadic bicompactum is a limit cardinal number. On the other hand, in (2), p. 234, it is shown that under the assumption (GCH)** there does not exist a wild dyadic bicompactum whose weight is an inaccessible number. However, the following is true

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* This theorem gives a partial answer to a question of V. V. Proizvolov.
** (GCH) –generalized continuum hypothesis, (CH) –continuum hypothesis.

Theorem 4. For every singular cardinal number τ there exists a wild dyadic bicompactum $S(\tau)$ of weight τ which, first, is an open image of D^τ and, second, an irreducible image of D^τ.*

We shall indicate only the construction of the bicompactum $S(\tau)$. Denote by $\mathfrak{D}_\tau = D^\tau \oplus (x)$ the topological sum (^{5}) of D^τ and an isolated point (x); by $\mathfrak{F}_\tau$ the Aleksandrov compactification of the topological sum of a countable number of copies of D^τ; by $\mathfrak{L}_\tau$ the space obtained from D^τ by contracting to a point some closed nowhere dense subset of type G_δ.

Lemma 3. If $\tau \geq \aleph_1$, then the spaces $(\mathfrak{D}_\tau)^{\aleph_0}$, $\mathfrak{F}_\tau$, and $\mathfrak{L}_\tau$ are homeomorphic to one another.

Lemma 4. The space $\mathfrak{D}_\tau$ is an open image of D^τ, and the space $\mathfrak{L}_\tau$ is an irreducible image of D^τ.

Let $\tau = \sum_{\alpha\in A} \mathfrak{n}_\alpha$, $\mathfrak{n}_\alpha < \tau$, $|A| < \tau$ be a singular number. Then

$$S(\tau) =$$

$$= \prod_{\alpha\in A} \mathfrak{F}_{\mathfrak{n}_\alpha}$$

is the required wild dyadic bicompactum of weight τ.

§ 3. **Dyadicity and hypotheses of set theory.** Let X be a topological space, $M \subset X$, and $x \in X$. We shall call the local power $\mathfrak{t}(x, M)$ of the set M relative to the point $x \in X$ the number $\mathfrak{t}(x, M) = \min |Ox \cap M|$**, if $Ox \in \mathfrak{F}$, where $\mathfrak{F}$ is the filter of all neighborhoods of the point x in X. A point $x \in X$ is called a point of extremal accumulation (^{1}) for the set M if $\mathfrak{t}(x, M) > \mathfrak{t}(y, M)$ for all $y \in X$ and $y \neq x$. In other words, a point $x \in X$ is a point of extremal accumulation for the set M if the function $\mathfrak{t}(x, M)$, defined on X, has an absolute maximum at the point x. The space X is called an χ-space (^{1}) if for every $M \subset X$ and every point $x \in \overline{M}$ there exists $M' \subset M$ for which x is a point of extremal accumulation in X.

An easy consequence of Theorem 9 from the author' s paper (^{3}) is the following

Theorem 5 (CH). Every nonmetrizable dyadic bicompactum topologically contains the Stone-Čech compactification βN of the natural numbers.***

Since βN is not an χ-space, while every closed subspace of an χ-space is an χ-space, we immediately obtain the following theorem, which is an answer to a question of A. V. Arhangel' skii and V. I. Ponomarev (^{1}).

Theorem 6 (CH). Every dyadic bicompactum which is an χ-space is metrizable.

One says that a space X satisfies Shanin's condition if every decreasing sequence, well ordered by the inclusion relation, of nonempty open subsets of X whose intersection is empty contains a countable cofinal subsequence. The following theorem is of interest in connection with the problem posed in (^{1}), p. 995. Let us note that we do not require Shanin' s condition to hold hereditarily.

Theorem 7 (CH). Every Hausdorff space X with the first axiom of countability, satisfying Shanin' s condition, is separable.

Proof. Since Shanin' s condition implies Suslin' s condition, X satisfies the first axiom of countability and Suslin' s condition. Hence, by a theorem of I. Juhász and A. Hajnal (^{6}), it follows that $|X| \leq \exp \aleph_0$. If $|X| \leq \aleph_0$, then all is proved. If $|X| = \aleph_1$, let us enumerate the points of X by all ordinals $\leq \omega_1$. Thus, let

$$X = (x_1, x_2, \dots, x_\alpha, \dots, \alpha < \omega_1).$$

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* This theorem is a solution of the problem posed in (^{2}), p. 235.
** $|A| = \operatorname{card} A$.
*** Let us note that in fact Theorem 5 is equivalent to the continuum hypothesis.

Put $F_\alpha = \{x_\beta,\ \beta < \alpha\}$, $\varphi_\alpha = \overline{F_\alpha}$. Suppose that X is not separable. Then for all $\alpha < \omega_1$ we have $U_\alpha = X \setminus \Phi_\alpha \neq \emptyset$. Hence the sequence $U_1 \supset U_2 \supset \cdots \supset U_\alpha \supset \cdots,\ \alpha < \omega_1$, is a decreasing well-ordered sequence of nonempty open sets, and

$$\bigcap_{\alpha<\omega_1} U_\alpha = \bigcap_{\alpha<\omega_1} (X \setminus \Phi_\alpha) = X \setminus \bigcup_{\alpha<\omega_1} \Phi_\alpha = X \setminus X = \emptyset.$$

It is easy to show that this sequence contains no countable cofinal subsequence, which contradicts Shanin' s condition. The theorem is proved.

Corollary (CH). *The Suslin continuum does not satisfy Shanin' s condition.*

The following theorem is a solution of the problem posed in (^{2}), p. 237.

Theorem 8 (GCH). *The cardinality of a dyadic bicompactum is either a power of two or is the sum of a countable number of smaller cardinals.*

Theorem 9. *The following two conditions are equivalent:*

a) (X is dyadic and $|X| \leq \exp \aleph_0$) $\Rightarrow$ (X is metrizable);

b) $\exp \aleph_0 < \exp \aleph_1$.

The following theorem is a strengthening of a theorem of A. S. Esenin-Vol' pin (^{7}).

Theorem 10. *If the weight of a dyadic bicompactum X has uncountable cofinal character, then there exists in X a subset M such that $\operatorname{int} \overline{M} \neq \emptyset$ and for every point $x \in M$ we have $\chi(x, X) = wX$.*

The following theorem gives an answer to a question posed to the author by M. Choban in connection with the work (8).

Theorem 11. *Every generalized metrizable dyadic bicompactum is metrizable.*

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Note: Figure translations are in progress. See original paper for figures.